Maximizing quadratic equations with an application to Bertrand duopoly

Bruno Salcedo[*]

Summer 2012

To identify players best responses it is necessary to maximize their expected utility $U_i(s_i, \theta_{-i})$. When the game is finite (and small) this is relatively easy, we can compute the expected utility for each pure strategy and compare them. However, this approach is not possible when the game is infinite. In such cases, the easiest approach usually involves calculus. Since this a calculus-free class, I've prepared this note that explains how to maximize a *concave* and *quadratic* expected utility function without using calculus.

1. Quadratic functions

A real function $f : \mathbb{R} \to \mathbb{R}$ is quadratic if it can be written as:

$$f(x) = -ax^2 + bx + c \tag{1}$$

for some real numbers $a, b, c \in \mathbb{R}$. If a is a positive number then f is a concave function. The graph of a quadratic and concave function is a *parabola* that opens downward. Furthermore, quadratic and concave functions are increasing before the *vertex* of the parabola and decreasing thereafter. Hence, they achieve their maximum value exactly at the vertex, see figure (1).

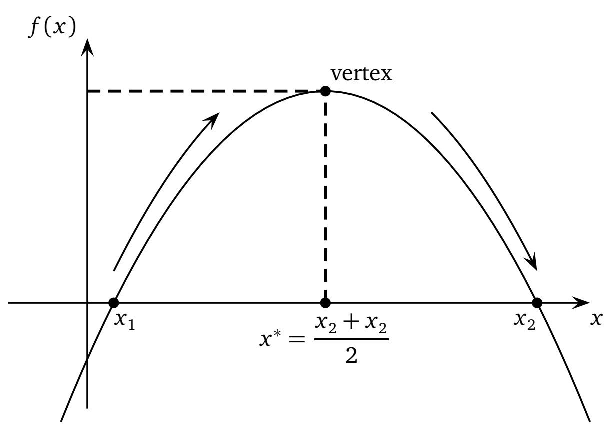

Figure 1: A concave quadratic function.

We say that a number $x_0 \in \mathbb{R}$ is a *root* of f if and only if $f(x_0) = 0$. Quadratic functions have at most two roots x_1, x_2 that can be obtained by using the quadratic formula:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4(-a)c}}{2(-a)} = \frac{b \pm \sqrt{b^2 + 4ac}}{2a} \tag{2}$$

[*]Personal website: personal.psu.edu/bxs5142/, email: bruno@psu.edu. These lecture notes where prepared as suplementary material for ECON402 at The Pennsylvania State University.

In all the examples we cover in class, our quadratic functions will have *exactly* two roots.[1] In this cases, the horizontal coordinate of the vertex is the average of the roots:[2]

$$x^* = \frac{x_1 + x_2}{2} = \frac{b + \sqrt{b^2 + 4ac}}{4a} + \frac{b - \sqrt{b^2 + 4ac}}{4a} = \frac{2b}{4a} = \frac{b}{2a} \tag{3}$$

Hence equation (3) offers a simple formula to find the value of x that maximizes a quadratic function $f(x)$ written as in equation (1).

Furthermore notice that:

$$x_1 + x_2 = \frac{b + \sqrt{b^2 + 4ac}}{2a} + \frac{b - \sqrt{b^2 + 4ac}}{2a} = \frac{2b}{2a} = \frac{b}{a} \tag{4}$$

$$\begin{aligned} x_1 \cdot x_2 &= \left(\frac{b + \sqrt{b^2 + 4ac}}{2a}\right)\left(\frac{b - \sqrt{b^2 + 4ac}}{2a}\right) = \frac{1}{4a^2}\left(b^2 - \left(\sqrt{b^2 + 4ac}\right)^2\right) \\ &= \frac{1}{4a^2}\left(b^2 - b^2 + 4ac\right) = \frac{4ac}{4a^2} = \frac{c}{a} \end{aligned} \tag{5}$$

This implies that we can rewrite f as:

$$\begin{aligned} f(x) &= -a\left(x - x_1\right)\left(x - x_2\right) \\ &= -a\left(x^2 + (x_1 + x_2)x + x_1 x_2\right) = -ax^2 + a \cdot \frac{b}{a} \cdot x + a \cdot \frac{c}{a} = -ax^2 + bx + c \end{aligned} \tag{6}$$

In some cases, the function that you want to maximize will be written in a form that resembles (6) instead of (1). In such cases you know that the roots of f are x_1 and x_2 and you can compute x^* by simply taking their average.

Summary. *Suppose that you want to find the value x^* that maximizes a concave and quadratic function $f : \mathbb{R} \to \mathbb{R}$.*

- *If f is written as:*

  $$f(x) = -ax^2 + bx + c$$

  with $a > 0$, then x^* is given by:

  $$x^* = \frac{b}{2a}$$

- *If f is written as:*

  $$f(x) = -a\left(x - x_1\right)\left(x - x_2\right)$$

  with $a > 0$ then x^* is given by:

  $$x^* = \frac{x_1 + x_2}{2}$$

I strongly recommend that you learn both formulas or learn how to derive both formulas because I won't know which one will be easier to use in the exams.

[1] A quadratic function may have only one root when $b^2 - 4ac = 0$ and no roots when $b^2 - 4ac < 0$. We will always have $b^2 - 4ac > 0$

[2] Notice that you could have found the same result using calculus. The necessary (and sufficient given the concavity of the function) condition for a maximum is $f'(x^*) = 0$. Using equation (1) we have $f'(x) = -2ax + b$ and thus $x^* = b/2a$.

2. Bertrand duopoly with imperfect substitutes

In this section we will use the techniques developed to find the best response functions for a Bertrand duopoly example. Suppose that there are two firms that we label 1 and 2 that produce different commodities. We label commodities with the same label as the producing firm (eg firm 1 produces commodity 1). Firms play a strategic form game in which they simultaneously and independently choose prices. Let p denote the price of commodity 1, q denote the price of commodity 2. The firms' payoffs correspond to their profits which, in turn depend on the quantities demanded.

The commodities produced are imperfect substitutes, so that the demand for one of the commodities *decreases* when the price of the other commodity *decreases*. Let $D_1(p,q)$ and $D_2(p,q)$ denote the demand functions. We assume that D_1 and D_2 are linear and are given by:

$$D_1(p,q) = M - p + kq \qquad D_2(p,q) = M - q + kp \tag{7}$$

for some positive numbers M and k. We also assume that firms have constant marginal costs $c > 0$. Firm's profits are then given by:

$$u_1(p,q) = \Big(p-c\Big)\cdot D_1(p,q) = \Big(p-c\Big)\Big(M-p+kq\Big) \tag{8}$$

$$u_2(p,q) = \Big(q-c\Big)\cdot D_2(p,q) = \Big(q-c\Big)\Big(M-q+kp\Big) \tag{9}$$

Given its beliefs, we can use the linearity of the expectation operator to write the expected payoff of firm 1 as a concave quadratic function:

$$U_1(p,\theta_2) = \mathbb{E}\Big[\Big(p-c\Big)\Big(M-p+kq\Big)\,\Big|\,\theta_1\Big] = -\Big(p-c\Big)\Big(p-M-k\bar{q}\Big) \tag{10}$$

where $\bar{q} = \mathbb{E}\left[\,q|\theta_2\,\right]$. Notice that this $U_1(p,\theta_2)$ is a concave and quadratic function written as in (6) with $a = 1$, $x_1 = c$ and $x_2 = M + k$. Hence we can use our formula to determine that firm 1's best response is given by:

$$BR_1(\theta_2) = \frac{1}{2}\Big(M + k\bar{q} + c\Big) = \frac{M+c}{2} + \frac{k}{2}\bar{q} \tag{11}$$

Similarly for firm 2 we have:

$$U_2(q,\theta_1) = -\Big(q-c\Big)\Big(q-M-k\bar{p}\Big) \qquad BR_2(\theta_1) = \frac{M+c}{2} + \frac{k}{2}\bar{p} \tag{12}$$

where $\bar{p} = \mathbb{E}\left[\,p|\theta_1\,\right]$.

3. A numerical example

Consider a Bertrand duopoly game with $M = 10$, $c = 2$ and $k = 1$, and suppose that the firms must choose prices in the interval $[0, 20]$. Using (11) and (12), the best response functions for each firm are given by:

$$BR_1(\theta_2) = \frac{M + c}{2} + \frac{k}{2}\bar{q} = 6 + \frac{1}{2}\bar{q} \tag{13}$$

$$BR_2(\theta_1) = \frac{M + c}{2} + \frac{k}{2}\bar{p} = 6 + \frac{1}{2}\bar{p} \tag{14}$$

Figure (**??**) illustrates the best response functions.

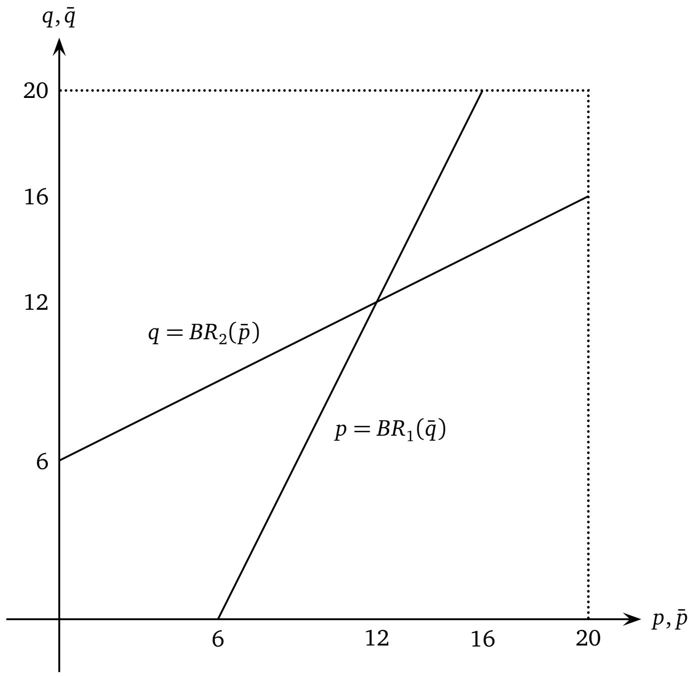

Figure 2: Best response functions

To illustrate the use of formula (3) we follow another approach to finding firm 1's best response function. By substituting with the values of the parameters in the expected payoff formulas we get:

$$\begin{aligned} U_1(p, \theta_2) &= -\Big(p - c\Big)\Big(p - M - k\bar{q}\Big) = -\Big(p - 2\Big)\Big(p - 10 - \bar{q}\Big) \\ &= -p^2 + 10p + p\bar{q} + 2p - 20 - 2\bar{q} = -p^2 + (12 + \bar{q})p - (20 + 2\bar{q}) \end{aligned} \tag{15}$$

this expression is similar to (1) with $a = 1$, $b = 12 + \bar{q}$ and $c = -(20 + 2\bar{q})$. Hence we can find the best response using formula (3):

$$BR_1(\theta_2) = \frac{b}{2a} = \frac{12 + \bar{q}}{2} = 6 + \frac{1}{2}\bar{q} \tag{16}$$