Common knowledge

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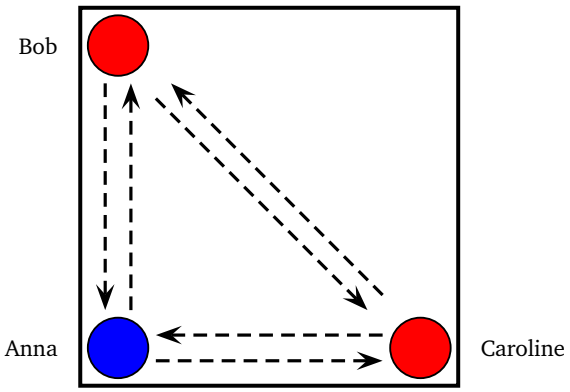
We say that a fact is mutually known if everybody knows it. We say that it is commonly known if everybody knows it, and, in addition, everybody knows that everybody knows it, everybody knows that everybody knows that everybody knows it, and so on and so forth. The following example illustrates that there is a big difference between mutual knowledge and common knowledge.

Anna, Bob and Caroline are ideal mathematicians: they have unbounded reasoning skills and they always speak the truth. They are sitting in opposing corners of a room without mirrors. Each one of them can see the other two but cannot see him/herself. For instance, Anna can see Bob and Caroline but not herself. We assume that all this information is common knowledge. Each one of them is wearing a hat: Anna is wearing a blue hat and Bob and Caroline are wearing red hats.



Case 1: mutual knowledge

Suppose that Daniel enters the room and tells the mathematicians that everybody is wearing either blue or red hats. Then he proceeds to ask them one by one *–Which color is your hat?* First he asks Anna, then Bob and then Caroline and in every case he gets the same answer: *–I don't know*.

Notice that it is common knowledge that everybody is wearing either a blue hat or a red hat. This is because this fact was publicly announced and everybody noticed that everybody heard it. This however does not imply that there has to be either a red hat or a blue hat, it might very well be the case (given the mathematicians information) that all hats are blue or all hats are red. Anna know that there are at least two red hats, because she can see Bob and Caroline's hats. But this doen't imply any information about her own hat. Similarly, Bob and Caroline know that there is at least one red hat and at least one blue hat but they can't infer anything about their own hats. Hence nobody is able to provide a definitive answer to Daniel's question. *Notice that it is mutual knowledge (everybody knows) that there is at least one red hat*.

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Case 2: common knowledge

Now suppose that Daniel enters the room and he tells the mathematicians that everybody is wearing either a blue or a red hat. In addition he tells them that there is at least one red hat in the room. The he proceeds as before asking them one by one –*Which color is your hat?* Anna and Bob answers as before: –*I don't know.* However, Caroline answers triumphant: –*My hat is red!*

The only difference between the two scenarios is that, in the second one, Daniel made the additional announcement that there is at least one red hat. However *this is something that everybody already knew*. The big difference is that, by making the announcement public, the existence of at least one red hat became common knowledge. In this case everybody knew that everybody knew that there was at least one red had. This is what allowed Caroline to deduce that her hat was red.

Caroline knew that Bob knew that there was at least one red hat. This implies that, if he had seen only blue hats he would have known that his own hat had to be the red one. Since he answered that he did not know the color of his own hat, it had to be the case that he was already seeing at least one red hat. That is, either Anna or Caroline hat to be wearing a red hat. Since Caroline could see that Anna's hat was blue, this meant that her own hat had to be red. *Notice that this line of thought was only possible because she knew that Bob knew that there was at least one red hat*.

In this example we did not use the full power of common knowledge. We only used what is commonly called second order mutual knowledge, i.e. everybody knows that everybody knows. However, it is possible to extend this example so that it only works if everybody knows that everybody knows that everybody knows (third order), or only if everybody knows that everybody knows that everybody knows that everybody knows (fourth order) or any arbitrary order of mutual knowledge.