Topics in Informational Economics 2
Games with Private Information and Selling Mechanisms

Watson §26-27, pages 312-333

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Private Information

- Previously we considered examples where players where completely uninformed about Nature's moves
- In many situations players hold private information
- Now we will consider examples in which Nature starts by making a random moves, and different players receive different signals that convey different information about these moves
- These environments are represented by Bayesian games

Bayesian Games

- A Bayesian (strategic form) game specifies:
    - The players involved in the environment
    - The choices available for each player
    - The different private signals (or types) that each player can receive
    - A joint distribution specifying the player's common prior beliefs about nature's choices (and thus about the private signals)
    - The expected payoff that each player will receive *conditional on the choice made by each agent and the signal received by each agent*

- You might (should?) think that the common prior assumption is unrealistic
- We are implicitly making two implicit assumptions:
    1. *That all players agree on the probability of unknown events*.– Aumann showed that if players start with common priors and their posterior are commonly known, then private information cannot generate disagreement
    2. *That each player knows exactly what other players believe*.– Mertens and Zamir showed that by choosing the right set of signals this assumption comes without loss of generality in most cases

Bayesian Games

Formal definition

Definition

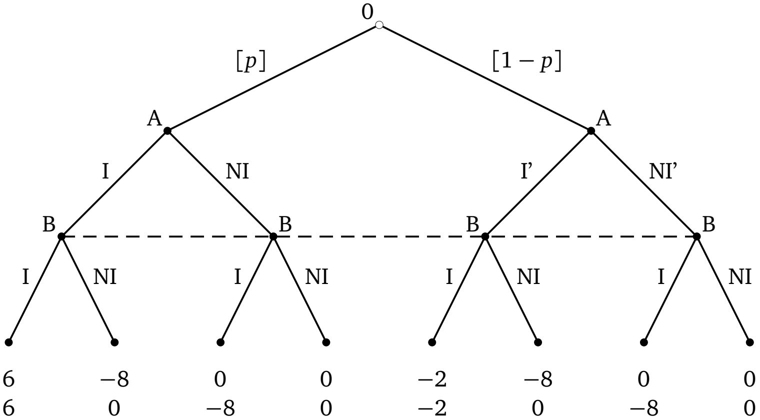
A strategic form Bayesian game is a mathematical object consisting of:

- A finite set of players $I = \{1, 2, \ldots, N\}$
- A set of strategies S_i for each player
- A set of types T_i for each player
- A prior distribution F over the the set of type profiles T
- A payoff function for each player $u_i : S \times T \to \mathbb{R}$

Definition

A Bayesian Nash equilibrium is a Nash equilibrium of a Bayesian game

Example: Risky investment with informed player



Example: Risky investment with informed player

	I	NI
I,I'	6 , 6	−8 , 0
I,NI'	6 , 6	−8 , 0
NI,I'	0 , −8	0 , 0
NI,NI'	0 , −8	0 , 0

Good (p)

	I	NI
I,I'	−2 , −2	−8 , 0
I,NI'	0 , −8	0 , 0
NI,I'	−2 , −2	−8 , 0
NI,NI'	0 , −8	0 , 0

Bad ($1 - p$)

Example: Risky investment with informed player

p=1/2

	I	NI
~~I,I'~~	~~2 , 2~~	~~−8 , 0~~
I,NI'	3 , −1	−4 , 0
~~NI,I'~~	~~−1 , −5~~	~~−4 , 0~~
NI,NI'	0 , −8	$\boxed{0 , 0}$

Example: Risky investment with informed player

p=3/4

	I	NI
~~I,I'~~	~~$4 , \underline{4}$~~	~~$-8 , 0$~~
I,NI'	$\boxed{\underline{4.5} , \underline{2.5}}$	$-2 , 0$
~~NI,I'~~	~~$-0.5 , -6.5$~~	~~$-2 , \underline{0}$~~
NI,NI'	$0 , -8$	$\boxed{\underline{0} , \underline{0}}$

Example: Battle for an Island

- The Row army and the Column army are considering invading an unoccupied island worth 6
- Each army can be strong (S) with probability 1/2 and weak with probability 1/2 and the strength of each army is not known by its opponent
- If only one army attacks the island it is captured and there are no costs involved
- If both armies attack then there is a battle and:
    - If the armies are of equal strength then no-one gets the island
    - If the armies have different types then the strong army wins the island
    - Fighting involves a cost of 4 for strong armies and 8 for weak armies

Example: Battle for an Island

	A	NA
A	$-4\,,\,-4$	$6\,,\,0$
NA	$0\,,\,6$	$0\,,\,0$

SS

	A’	NA’
A	$2\,,\,-8$	$6\,,\,0$
NA	$0\,,\,6$	$0\,,\,0$

SW

	A	NA
A’	$-8\,,\,2$	$6\,,\,0$
NA’	$0\,,\,6$	$0\,,\,0$

WS

	A’	NA’
A’	$-8\,,\,-8$	$6\,,\,0$
NA’	$0\,,\,6$	$0\,,\,0$

WW

Example: Battle for an Island

	A,A'	A,NA'	NA,A'	NA,NA'
A,A'	$-1.5\,,\,-1.5$	$0\,,\,-0.5$	$1.5\,,\,-4$	$\boxed{\underline{6}\,,\,\underline{0}}$
A,NA'	$-0.5\,,\,0$	$\boxed{\underline{0.5}\,,\,\underline{0.5}}$	$\underline{2}\,,\,-0.5$	$3\,,\,0$
~~NA,A'~~	$\cancel{-4\,,\,1.5}$	$\cancel{-0.5\,,\,\underline{2}}$	$\cancel{-0.5\,,\,-0.5}$	$\cancel{3\,,\,0}$
NA,NA'	$\boxed{\underline{0}\,,\,\underline{6}}$	$0\,,\,3$	$0\,,\,3$	$0\,,\,0$

Optimal selling mechanisms

- Suppose that you wanted to sell an object and you knew exactly how much each person in the world is willing to pay for it
- The optimal selling mechanism in this case is very simple: *Sell it to the person who values it the most charging the maximum amount that he/she is willing to pay for it*
- This is what happens in perfectly competitive markets, hence the welfare theorems

Theorem (First undamental welfare theorem)

The allocation of a competitive general equilibrium is Pareto efficient

Theorem (First undamental welfare theorem)

Every Pareto efficient allocation can result from a general equilibrium with lump-sum transfers

- There are many reasons why markets may fail: one of the most important ones is information

Selling to an informed buyer

- Anna wants to sell a painting to Bob
- Bob knows the value v of the painting but Anna knows very little about Art and she values the painting 0
- She believes that the value of the painting is uniform $[0, 1]$ so that trade is always efficient
- If she asks Bob how much the painting is worth, Bob has no incentives to tell the truth, he will always want to say that it is worthless to get it for free

- Suppose (WLOG) that Anna makes a take it or leave it offer p
- Bob will accept the price if and only if $v \leq p$
- Anna's expected payoff is then:

$$U_A(p) = p \cdot \text{Prob}(v \geq p) = p(1 - p) = p - p^2$$

- And the optimal price is then $p^* = 1/2$ that results in $U_A^* = 1/4$

- Notice that this optimal price is not efficient: when $v < 1/2$ it would be efficient for Anna to sell the painting to Bob

Selling to many buyers

- Now suppose that Anna has many potential buyers
- Now she has to chose not only a selling price but also the person that will receive the painting

- There are many mechanisms that she could use to determine this, without loss of generality we can restrict our attention too direct selling mechanisms or auctions in which:
  1. Anna announces an allocation rule and a transfer rule that depend on the valuation of each potential buyer
  2. Each buyer announces his/her valuation (simultaneously and independently) and the object is allocated and transfers take place according to the announced rules
- This important theorem is known as the Revelation Principle and is very general (goes beyond allocation problems)

- In this class assume that there are only two potential buyers with private and independent values distributed uniformly on $[0, 1]$
- In this case, there are mechanisms that allocate the object efficiently (to the buyer with the highest valuation) but the optimal (revenue maximizing) mechanism is inefficient

First price auction

- Suppose that Anna uses a first-price sealed bid auction:
  - Players simultaneously and independently submit bids b_i
  - The player with the highest bid gets the object and pays his/her bid
- The payoff for player i given a type profile v and bid profile b is:

$$u_i(v,b) = \begin{cases} v_i - b_i & \text{if} \quad v_i > v_{-i} \\ 0 & \text{eoc} \end{cases}$$

- *Guess*: in equilibrium bids are linear functions of values, i.e. $b_i = a v_i$
- In this case, the expected payoff for player i is:

$$U_i(v_i, b_i | s_{-i}) = \left(v_i - b_i\right) \text{Prob}\left(a v_{-i} \leq b_i\right) = \left(v_i - b_i\right) \frac{b_i}{a}$$

- And therefore the best response is to choose $b_i^* = v_i/2$
- Since this best response is linear it follows that the bid functions $b_i(v_i) = v_i/2$ are a BNE

Second price auction

- Suppose that Anna uses a first-price sealed bid auction:
  - Players simultaneously and independently submit bids b_i
  - The player with the highest bid gets the object and pays *the second highest bid*
- Notice that choosing $b_i = v_i$ is weakly dominant because:
  - If you bid $b_i > v_i$ and you win the object you will make a loss
  - If you bid $b_i < v_i$ and you win the object then you would have also won with $b_i = v_i$ and you will pay the same
- Hence the bid functions $b_i(v_i) = v_i$ are a BNE

Revenue equivalence theorem

- Both the first price auction and the second price auction are efficient: they allocate the object to the buyer with the higher valuation
- Also, they both generate the same revenue $1/3$ with two players or $N-1/N+1$ for N players (requires calculus)
- This is not a coincidence, it follows from the revenue equivalence theorem:

Theorem

If two mechanisms have the same allocation rule, and the same expected payment conditional on $v_i = 0$, then they generate the same expected revenue

Reservation prices

- The first and second price auctions are not optimal fro the seller
- The optimal mechanism is actually a second (or first) price auction with a reservation price r, think of the seller as anther potential buyer that always bids r
- In the second price auction it is still a BNE to bid $b_i = v_i$, however the expected revenue is different now because:
  - When all players have valuations below r the seller will keep the object
  - When one player has a valuation below r and the other player has a valuation above r, the payment will be r instead of the minimum value
- The optimal reservation price turns out to be $r^* = 1/2$ which results in revenue $5/12$
- As in the case with a single buyer, the optimal mechanisms is not efficient: sometimes the seller keeps the object even tough there are buyers with higher valuations

Two sided private information