Beliefs, expected utility and best responses

Watson §4 pages 38-40 & §6 pages 50-52

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Example: uggs or rain boots

Rational choice under uncertainty

- Emma would like to wear her ugg boots today but she is concerned that it might rain
- If it does rain she would prefer to wear her rain boots
- The problem is that she is *uncertain* about whether it is going to rain
- She believes that it is going to rain with probability $p \in (0, 1)$

	No Rain $[1-p]$	Rain $[p]$
Ugg boots	10	-5
Rain boots	4	6

Example: uggs or rain boots

Expected utility

- Emma's expected utility from wearing her ugg boots is:

$$U\left(\text{Ugg boots}, p\right) = 10(1-p) - 5p = 10 - 15p$$

- Emma's expected utility from wearing her rain boots is:

$$U\left(\text{Rain boots}, p\right) = 4(1-p) + 6p = 4 + 2p$$

- Emma will choose to wear her ugg boots if and only if:

$$U\left(\text{Ugg boots}, p\right) \geq U\left(\text{Rain boots}, p\right) \quad \Leftrightarrow \quad p \leq \frac{6}{17} \approx 35\%$$

Rational choice under uncertainty

- Uncertainty means lack of information
- We say that a player is uncertain about an event if he doesn't know whether it is true or not
- We use the word "beliefs" to mean probability functions that represent the likelihood of each possibility
- We assume that players always maximize their expected utility given their beliefs

Beliefs

- In a strategic form game, since choices are independent, each player is uncertain about the strategies chosen by his opponents

Definition

Given a strategic form game, a belief for player $i \in I$ is a *probability distribution* θ_{-i} over his/her opponent's strategy sets

- We interpret $\theta_{-i}(s_{-i})$ as a measure of the likelihood that player i assigns to his/her opponents choices corresponding to s_{-i}
- When S_{-i} is finite and has N elements, then a belief for player i is just a vector consisting of N numbers between 0 and 1 that add up to 1.

Example: Battle of the sexes

Beliefs

	Football	Opera
Football	5 , 1	0 , 0
Opera	0 , 0	1 , 5

- A belief for Mike is a pair of numbers $\left(\theta_N(F), \theta_N(O)\right)$ between 0 and 1 such that $\theta_N(F) + \theta_N(O) = 1$
- We simplify the notation by using $p = \theta_N(F)$ and $(1 - p) = \theta_N(O)$
- p is the probability that Mike assigns to Nancy going to the football game and $(1 - p)$ is the probability that Mike assigns to Nancy going to the Opera

Expected utility

- Given i's beliefs θ_{-i} about his/her opponent's behavior we can define his/her expected payoff or expected utility from choosing a strategy s_i:

$$U_i(s_i, \theta_i) = \mathbb{E}\left[\, u_i(s_i, s_{-i}) | \theta_i \,\right]$$

- For finite games, expected utility is jut the weighted sum of the payoffs that i would get from different choices of his/her opponents weighted by how likely he/she belief that these choices are:

$$U_i(s_i, \theta_i) = \sum_{s_{-i} \in S_{-i}} \theta_{-i}(s_{-i}) u_i(s_i, s_{-i})$$

Example: Battle of the sexes

Expected utility

	Football $[p]$	Opera $[1-p]$
Football	5 , 1	0 , 0
Opera	0 , 0	1 , 5

- Given his beliefs, Mike's expected utility for going to the football game is:

$$U_M(\text{Football}, p) = 5 \cdot p + 0 \cdot (1-p) = 5p$$

- His s expected utility for going to the opera is:

$$U_M(\text{Opera}, p) = 0 \cdot p + 1 \cdot (1-p) = 1-p$$

Example: Battle of the sexes

Expected utility

	Football $[p]$	Opera $[1-p]$
Football $[q]$	5 , 1	0 , 0
Opera $[1-q]$	0 , 0	1 , 5

- Given her beliefs, Nancy's expected utility for going to the football game is:

$$U_N(\text{Football}, q) = 1 \cdot q + 0 \cdot (1-q) = q$$

- His s expected utility for going to the opera is:

$$U_N(\text{Opera}, q) = 0 \cdot q + 5 \cdot (1-q) = 5 - 5q$$

Example: A 4×4 game

Expected utility

	A $[\theta_2(A)]$	B $[\theta_2(B)]$	C $[\theta_2(C)]$	D $[\theta_2(D)]$
a $[\theta_1(a)]$	7 , 9	4 , 5	6 , 4	2 , 2
b $[\theta_1(b)]$	2 , 5	5 , 2	8 , 6	9 , 8
c $[\theta_1(c)]$	5 , 4	2 , 1	1 , 3	4 , 5
d $[\theta_1(d)]$	1 , 8	4 , 7	4 , 4	1 , 9

$$U_1(a, \theta_2) = 7\theta_2(A) + 4\theta_2(B) + 6\theta_2(C) + 2\theta_2(D)$$
$$U_1(c, \theta_2) = 5\theta_2(A) + 2\theta_2(B) + \theta_2(C) + 4\theta_2(D)$$
$$U_2(B, \theta_1) = 5\theta_1(a) + 2\theta_1(b) + \theta_1(c) + 7\theta_1(d)$$
$$U_2(D, \theta_1) = 2\theta_1(a) + 8\theta_1(b) + 5\theta_1(c) + 9\theta_1(d)$$

Example: Uneven thumbs

Expected utility

	Up $[\theta_2(\text{Up})]$	Down $[\theta_2(\text{Up})]$
Up	$0, 0, 0$	$1, -1, 1$
Down	$-1, 1, 1$	$1, 1, -1$

Up $[\theta_3(\text{Up})]$

	Up $[\theta_2(\text{Up})]$	Down $[\theta_2(\text{Up})]$
Up	$1, 1, -1$	$-1, 1, 1$
Down	$1, -1, 1$	$0, 0, 0$

Down $[\theta_3(\text{Down})]$

$$U_1(\text{Up}, \theta_{-1}) = \theta_2(\text{Up})\theta_3(\text{Down}) + \theta_2(\text{Down})\theta_3(\text{Up}) - \theta_2(\text{Down})\theta_3(\text{Down})$$

$$U_1(\text{Down}, \theta_{-1}) = \theta_2(\text{Up})\theta_3(\text{Down}) + \theta_2(\text{Down})\theta_3(\text{Up}) - \theta_2(\text{Up})\theta_3(\text{Up})$$

Example: Bertrand competition

Expected utility

- Recall our Bertrand example with firms $\{1,2\}$ choosing prices $p,q \in [0,10]$ and payoff functions:

$$u_1(p,q) = -p^2 + \left(12 + \frac{1}{2}q\right)p - \left(20 + q\right)$$
$$u_2(p,q) = -q^2 + \left(12 + \frac{1}{2}p\right)q - \left(20 + p\right)$$

- Firm 1's expected utility is given by:

$$U_1(p,\theta_1) = \mathbb{E}\left[-p^2 + \left(12 + \frac{1}{2}q\right)p - \left(20 + q\right) \,\middle|\, \theta_1 \right]$$
$$= -p^2 + \left(12 + \frac{1}{2}\bar{q}\right)p - \left(20 + \bar{q}\right)$$

where $\bar{q} = \mathbb{E}\left[\, q|\theta_2 \,\right]$

Best responses

- Recall that out notion of rationality assumes that players are expected utility maximizers
- Given his/her beliefs, a player should choose a strategy s_i that maximizes his/her expected utility
- We call such actions best responses

Definition

A strategy $s_i \in S_i$ is a best response to a belief θ_i if and only if it maximizes i's expected utility given θ_{-i}, i.e. if and only if:

$$U_i(s_i, \theta_{-i}) \geq U_i(s_i', \theta_{-i})$$

for every other strategy $s_i' \in S_i$

- We use the symbol $BR_i(\theta_{-i}) \subseteq S_i$ to denote the set of strategies for i that are best responses to θ_i

Example: Battle of the sexes

Best responses

- Mancy's expected utility functions in the Battle of the Sexes example are given by:

$$U_M(\text{Football}, p) = 5p \qquad U_M(\text{Opera}, p) = 1 - p$$

- Going to the football game is a best response if and only if:

$$U_M(\text{Football}, p) \geq U_M(\text{Opera}, p) \quad \Leftrightarrow \quad p \geq \frac{1}{6}$$

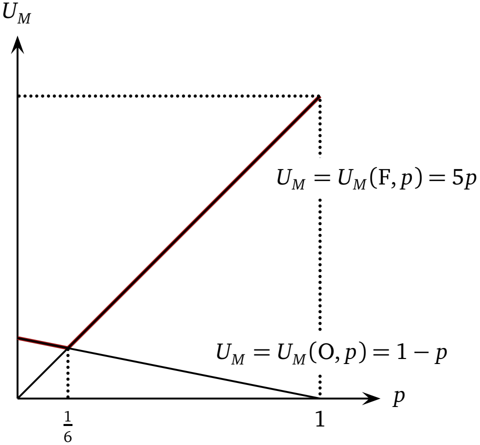
- Going to the opera game is a best response if and only if:

$$U_M(\text{Football}, p) \leq U_M(\text{Opera}, p) \quad \Leftrightarrow \quad p \leq \frac{1}{6}$$

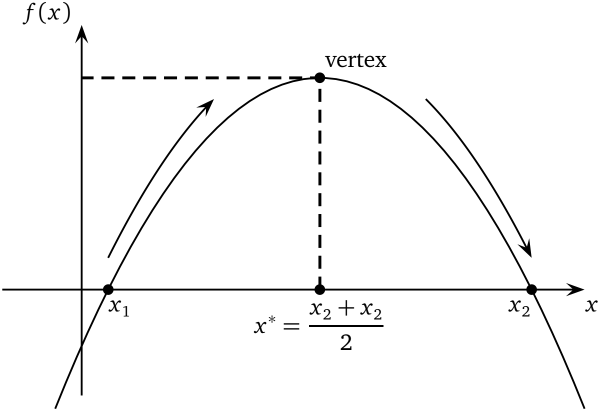
- Mike is indifferent between going to the opera of to the football game when $p = \frac{1}{6}$

Example: Battle of the sexes

Best responses



Maximizing quadratic functions



[See the corresponding lecture note for further details PDF]

Example: Bertrand competition

Best responses

- In our Bertrand example, firm 1's expected utility is given by:

$$U_1(p, \theta_1) = -p^2 + \left(12 + \frac{1}{2}\bar{q}\right) p - \left(20 + \bar{q}\right)$$

- As a function of p (taking θ_1 as a parameter) it is a parabola that opens down and has a unique best response:

$$p = 6 + \frac{1}{4}\bar{q}$$

- See the corresponding lecture notes for further details PDF

Example: Bertrand competition

Best responses