# Solution Concepts 3
# Nash equilibrium in pure strategies
### Watson §9-§10, pages 89-100

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# Rationalizability vs equilibrium

- By assuming that there is common knowledge of rationality, we have concluded that players will choose rationalizable strategies
- This prediction has two criticisms:
  1. In most cases it is not very informative
  2. It allows players to have erroneous beliefs
- By assuming that players beliefs are correct (i.e. if player 1 has beliefs $\theta_2$ about 2's behavior then 2 makes choices according to $\sigma_2 = \theta_2$) we obtain different notions of equilibrium
- In this slides we only consider *Nash equilibrium in pure strategies*

# Correct beliefs

- *Why would we assume that players have correct beliefs?*
  1. Communication.– If players communicate with each other prior to playing the game they might agree to follow some strategies
  2. Learning.– If players interact repeatedly they might learn from experience how to predict their opponents behavior
  3. Adaptation.– If players follow simple adaptive rules, behavior can also converge to something that looks like an equilibrium
  4. Institutions.– Institutions/mediators might help to coordinate players expectations
  5. Focal points.– Some rationalizable strategies might be justifiable by simple logical arguments

# Nash equilibrium in pure strategies

## Communication and self-enforcing agreements

- Suppose that the players gather to discuss and agree on playing according to some strategy profile $s \in S$ specifying a pure strategy for each player (no mixing for now)
- After that, players go different ways ant they choose strategies simultaneously and independently
- Suppose that player $i$ thinks that his/her opponents will not deviate from the agreed strategy profile, i.e. that they will choose the strategies in $s_{-i}$
- Then $i$ will be willing to choose strategy $s_i$ if and only if it is a best response to $s_{-i}$, i.e. if and only if $s_i \in BR_i(s_{-i})$
- In this case $i$ can not **strictly** benefit from **unilaterally** deviating from the intended strategy profile
- If no players have strict incentives to deviate **unilaterally** then we say that $s$ is a Nash equilibrium in pure strategies

# Nash equilibrium in pure strategies

## Definition

Given a strategic form game, a Nash equilibrium *in pure strategies* is a strategy profile $s \in S$ such that no player can **strictly** gain from deviating **unilaterally**, i.e. such that:

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

for every player $i$ and every alternative strategy $s_i' \in S_i$

- Equivalently, a Nash equilibrium is a profile of strategies which are best responses to each other, i.e. a strategy profile $s \in S$ such that $s_i \in BR_i(s_{-i})$ for every player $i$
- In a two player game (represented by a payoff matrix) a pair of strategies is a Nash equilibrium if player 1 is maximizing his/her payoff along the corresponding *column* and player 2 is maximizing his/her payoff along the corresponding *row*

# Example: Battle of the Sexes

## Nash equilibria

| | Football | Opera |
|---|---|---|
| Football | 5 , 1 | 0 , 0 |
| Opera | 0 , 0 | 1 , 5 |

- To find Nash equilibrium of a finite game one can start by highlighting the best response payoffs for each player
- If a cell in the matrix has all payoffs highlighted then it is a Nash equilibrium

# Rationalizability vs Nash equilibrium

- If we assume that:
  1. Players are rational
  2. Players are making deterministic choices (no mixed strategies)
  3. Players have correct beliefs about their opponents' behavior (they know what their opponents are going to choose)

  then we can predict that they sill play some Nash equilibrium

- Nash equilibria are *joint* predictions specifying strategies for *all* players
- Rationalizability makes *individual* predictions for *each* player

**Theorem**

*Every strategy in a Nash equilibrium is rationalizable*

**Theorem**

*If there is a unique rationalizable strategy for each player, then these strategies conform a Nash equilibrium*

# Example: A $4 \times 4$ game

## Best responses

| | a | b | c | d |
|---|---|---|---|---|
| w | $0, \underline{7}$ | $2, 5$ | $\underline{7}, 0$ | $0, 1$ |
| x | $5, 2$ | $\underline{3}, \underline{3}$ | $5, 2$ | $0, 1$ |
| y | $\underline{7}, 0$ | $2, 5$ | $0, \underline{7}$ | $0, 1$ |
| z | $0, \underline{0}$ | $0, -2$ | $0, \underline{0}$ | $\underline{10}, -1$ |

# Example: A $4 \times 4$ game

## Nash equilibrium and rationalizable strategies

| | a | b | c | d |
|---|---|---|---|---|
| w | 0 , 7 | 2 , 5 | 7 , 0 | 0 , 1 |
| x | 5 , 2 | 3 , 3 | 5 , 2 | 0 , 1 |
| y | 7 , 0 | 2 , 5 | 0 , 7 | 0 , 1 |
| z | 0 , 0 | 0 , −2 | 0 , 0 | 10 , − 1 |

# Example: classic $2 \times 2$ examples

## Best responses

| | Full | Empty |
|---|---|---|
| Full | $3, 3$ | $0, \underline{5}$ |
| Empty | $\underline{5}, 0$ | $\underline{2}, \underline{2}$ |

| | Continue | Swerve |
|---|---|---|
| Continue | $0, 0$ | $\underline{5}, \underline{1}$ |
| Swerve | $\underline{1}, \underline{5}$ | $2, 2$ |

| | GCS | PS |
|---|---|---|
| GCS | $\underline{1}, \underline{1}$ | $0, 0$ |
| PS | $0, 0$ | $\underline{2}, \underline{2}$ |

| | Press | Don't press |
|---|---|---|
| Press | $3, 1$ | $\underline{0}, \underline{5}$ |
| Don't press | $\underline{6}, -2$ | $-1, \underline{-1}$ |

# Example: classic $2 \times 2$ examples

## Nash and rationalizability

| | Full | Empty |
|---|---|---|
| Full | 3 , 3 | 0 , 5 |
| Empty | 5 , 0 | 2 , 2 |

| | Continue | Swerve |
|---|---|---|
| Continue | 0 , 0 | 5 , 1 |
| Swerve | 1 , 5 | 2 , 2 |

| | GCS | PS |
|---|---|---|
| GCS | 1 , 1 | 0 , 0 |
| PS | 0 , 0 | 2 , 2 |

| | Press | Don't press |
|---|---|---|
| Press | 3 , 1 | 0 , 5 |
| Don't press | 6 , −2 | 1 , 1 |

# Example: rock paper scissors

Not every game has a Nash equilibrium in pure strategies

| | Rock | Paper | Scissors |
|---|---|---|---|
| Rock | 0 , 0 | −1 , 1 | 1 , −1 |
| Paper | 1 , −1 | 0 , 0 | −1 , 1 |
| Scissors | −1 , 1 | 1 , −1 | 0 , 0 |

# Example: Cournot competition

## Best responses

- Consider a Cournt duopoly game with two firms 1 and 2 choosing quantities $q_1, q_2 \in [0, 50]$, with constant marginal costs $c = 10$ and inverse demand function:

$$P(q_1, q_2) = 100 - q_1 - q_2$$

- Payoffs are given by:

$$u_1(q_1, q_2) = (90 - q_2 - q_1)q_1 \qquad u_2(q_1, q_2) = (90 - q_1 - q_2)q_2$$

- Best responses to pure strategies are given by:

$$BR_1(q_2) = 45 - \frac{1}{2}q_2 \qquad BR_2(q_1) = 45 - \frac{1}{2}q_1$$

# Example: Cournot competition

## Nash equilibria

- A pure strategy Nash equilibrium for this Cournot example is a pair of quantities $(q_1, q_2)$ that are mutual best responses, i.e such that:

$$q_1 = BR_1(q_2) \qquad q_2 = BR_2(q_1)$$

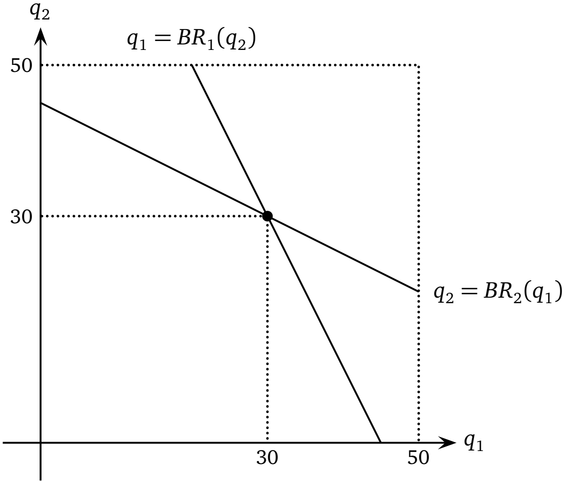
- Using our formula for best responses this is equivalent to:

$$\begin{aligned}
& q_1 = 45 - \frac{1}{2}q_2 \qquad q_2 = 45 - \frac{1}{2}q_1 \\
\Rightarrow \quad & q_2 = 45 - \frac{1}{2}\left(45 - \frac{1}{2}q_2\right) = 45 - 22.5 + \frac{1}{4}q_2 = 22.5 + \frac{1}{4}q_2 \\
\Rightarrow \quad & \frac{5}{4}q_2 = 22.5 \qquad \Rightarrow \qquad q_2 = \frac{4 \cdot 22.5}{5} = 30 \\
\Rightarrow \quad & q_1 = 45 - \frac{1}{2}30 = 45 - 15 = 30
\end{aligned}$$

- So the game has a unique Nash equilibrium in pure strategies: $(30, 30)$
- Recall that this was the *unique* rationalizable strategy profile

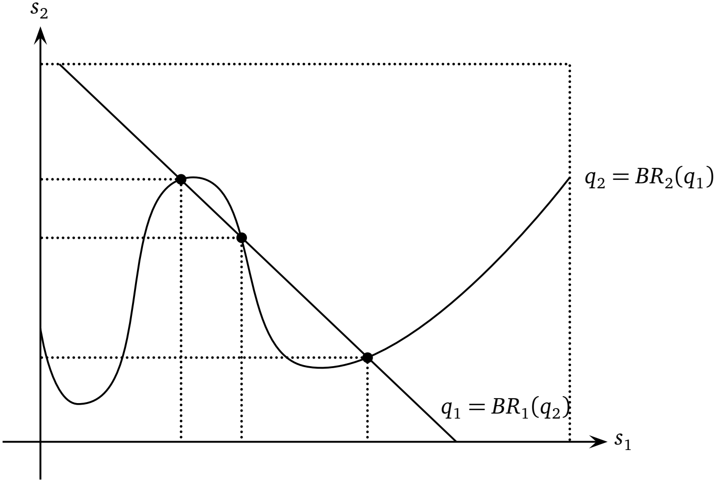
# Example: Cournot competition

## Nash equilibrium



# Example: a continuous two player game

## Best responses and Nash equilibrium



# Example: location game

## Nash equilibrium

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 35, 35 | 10, 60 | 15, 55 | 20, 50 | 25, 45 | 30, 40 | 35, 35 |
| 2 | 60, 10 | 35, 35 | 20, 50 | 25, 45 | 30, 40 | 35, 35 | 40, 30 |
| 3 | 55, 15 | 50, 20 | 35, 35 | 30, 40 | 35, 35 | 40, 30 | 45, 25 |
| 4 | 50, 20 | 45, 25 | 40, 30 | 35, 35 | 40, 30 | 45, 25 | 50, 20 |
| 5 | 45, 25 | 40, 30 | 35, 35 | 30, 40 | 35, 35 | 50, 20 | 55, 15 |
| 6 | 40, 30 | 35, 35 | 30, 40 | 25, 45 | 20, 50 | 35, 35 | 60, 10 |
| 7 | 35, 35 | 30, 40 | 25, 45 | 20, 50 | 15, 55 | 10, 60 | 35, 35 |