Solution Concepts 4
Nash equilibrium in mixed strategies
Watson §11, pages 123-128

Bruno Salcedo

The Pennsylvania State University

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Mixing strategies

- In a strictly competitive situation players have incentives to prevent their opponents from predicting their choices
  - Examples: rock paper scissors, military tactics, poker
- One way of remaining “unpredictable” is to randomize your choices

Definition

A mixed strategy for player i is a probability distribution σ_i over his/her strategies

- See the slides on dominance and best responses (S4) or section §5 in the textbook for more details.
- We don’t think of actual explicit randomization (eg rolling a dice to make a choice) but rather implicit randomization (eg basing your choices on “feelings” or unpredictable introspective processes)
- We use the adjective “pure” to talk about non-mixed strategies. A pure strategy is equivalent to the mixed strategy that plays it for sure

Nash equilibrium in mixed strategies

- When players randomize, we can compute expected utility:

$$U_i(\sigma_i, \sigma_{-i}) = \mathbb{E}\left[u_i(s_i, s_{-i}) \,\middle|\, \sigma_i, \sigma_{-i} \right]$$
$$= U_i(\sigma_i, \sigma_{-i}) = \sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \quad \text{(for finite games)}$$

- The notions of rationality, rationalizability, best responses and Nash equilibrium remain unchanged

Definition

Given a strategic form game, a Nash equilibrium is a (pure or mixed) strategy profile σ such that no player can **strictly** gain from deviating **unilaterally**, i.e. such that:

$$U_i(\sigma_i, \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i})$$

for every player i and every alternative strategy σ_i'

Example: Rock Paper Scissors

	Rock	Paper	Scissors
Rock	$0\,,\,0$	$-1\,,\,1$	$1\,,\,-1$
Paper	$1\,,\,-1$	$0\,,\,0$	$-1\,,\,1$
Scissors	$-1\,,\,1$	$1\,,\,-1$	$0\,,\,0$

- Claim: both players randomizing according to $(1/3, 1/3, 1/3)$ is a Nash equilibrium
- If a player uses this strategy his/her opponent's expected payoff *for any strategy* is 0
- Thus there are no incentives to deviate unilaterally

Computing equilibria in mixed strategies

Theorem

If a mixed strategy σ_i is a best response to σ_{-i} then so are all the strategies that are mixed with positive probability

- This means that, if a player is willing to randomize, it must be the case that he/she is *indifferent* between all the strategies over which he is randomizing
- To find Nash equilibria in mixed strategies we do the following:
  1. "Guess" the pure strategies that will be mixed (start by eliminating strategies that are not rationalizable)
  2. For each player i, look for a mixed strategy for $-i$ that makes i be indifferent between the strategies that he/she is mixing

Example: A 2×2 game

Row's expected utility

		Col L $[p]$	Col R $[1-p]$
Row	U $[q]$	3 , 3	5 , 8
	D $[1-q]$	1 , 2	6 , 1

- Given p, row's expected utility for each pure strategy is:

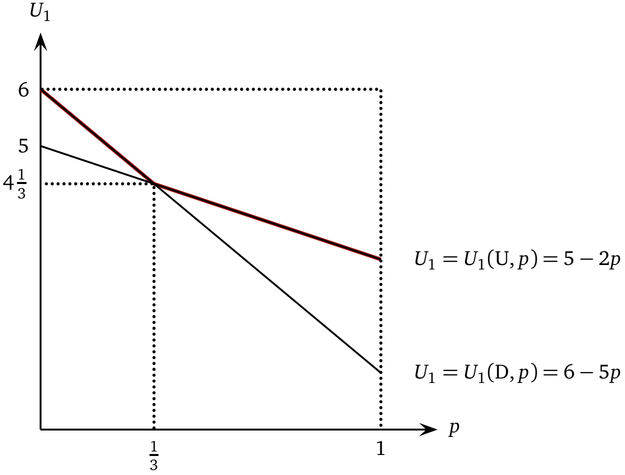
$$U_1(U,p) = 3p + 5(1-p) = 5 - 2p$$
$$U_1(D,p) = 1p + 6(1-p) = 6 - 5p$$

- Row is thus indifferent between U and D if and only if:

$$U_1(U,p) = U_1(D,p) \quad \Leftrightarrow \quad 5 - 2p = 6 - 5p \quad \Leftrightarrow \quad p = \frac{1}{3}$$

Example: A 2×2 game

Row's best responses



Example: A 2×2 game

		Col	
		L $[p]$	R $[1-p]$
Row	U $[q]$	3 , 3	5 , 8
	D $[1-q]$	1 , 2	6 , 1

- Given q, Col's expected utility for each pure strategy is:

$$U_2(L,q) = 3q + 2(1-q) = 2 - q$$
$$U_2(R,q) = 8q + 1(1-q) = 7q - 1$$

- Col is thus indifferent between L and R if and only if:

$$U_2(L,q) = U_2(R,q) \quad \Leftrightarrow \quad 2 - q = 7q - 1 \quad \Leftrightarrow \quad q = \frac{1}{6}$$

Example: A 2×2 game

		Col	
		L $[p]$	R $[1-p]$
Row	U $[q]$	3 , 3	5 , 8
	D $[1-q]$	1 , 2	6 , 1

- We then have found a mixed equilibrium in pure strategies:

$$\sigma_1 = \left(\frac{1}{6}, \frac{5}{6}\right)$$
$$\sigma_2 = \left(\frac{1}{3}, \frac{2}{3}\right)$$

Why bother making opponent be indifferent?

- It might not seem intuitive that a player randomizes with the exact probabilities that make his/her opponent be indifferent.
- Recall: *making an opponent indifferent is not the intention of the player*, the player simply wants to maximize his expected utility
- The definition and motivation of Nash equilibrium is only that players want to maximize their expected utility, and their beliefs are in equilibrium (there are no profitable unilateral deviations)
- The fact that the corresponding strategies must make players indifferent is a result

Example: A 4×4 game

	a	b	c	d
w	0 , 9	0 , 4	0 , 2	0 , 6
x	2 , 1	9 , 3	1 , 7	2 , 2
y	7 , 1	0 , 0	3 , 5	0 , 2
z	2 , 1	1 , 8	4 , 0	1 , 4

- Using iterated dominance we end up with a 2×2 game
- Let p be the probability of b and $1-p$ the probability of c, for indifference we must have:

$$9p + (1-p) = p + 4(1-p) \quad \Leftrightarrow \quad p = \frac{3}{11}$$

- Let q be the probability of x and $1-q$ the probability of z, for indifference we must have:

$$3q + 8(1-q) = 7q + 0(1-q) \quad \Leftrightarrow \quad q = \frac{2}{3}$$

Existence of equilibrium

Theorem

*Every **finite** strategic form game has **at least** one Nash equilibrium*

Theorem

Generically, finite strategic form games have an odd number of Nash equilibria

Example: A 2 × 2 game

Existence of equilibria