Solution Concepts 5
Subgame perfect equilibrium
Watson §14-§15, pages 159-175 & §19 pages 214-225

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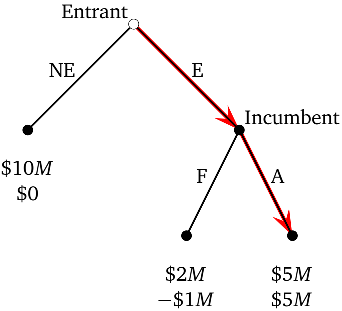
If you want to pass this class you have to take all the money you have in your wallet and bring it to me. Those of you that don't give me any money will automatically fail the class.

Incredible threats

- Hopefully, most of you would not give me any money because: *you don't really believe that I would carry out my threat*
- If I carried out my threat you could report me to my supervisor and I would certainly loose my job
- Conditional on you not giving me any money, it is not rational for me to actually give you a failing grade
- This argument depends on the *dynamic* structure of the game and might be lost if we only look to the strategic form game
- You paying up and me failing you unless you pay, might actually be a Nash equilibrium of the strategic form game
- This suggests that to make sensible predictions we might have to look at the dynamic structure of the extensive form game

Example: Entry deterrence

Incredible threats



	E	NE
F	2, −1	10, 0
A	5, 5	10, 0

There are two Nash equilibria in pure strategies, but (F,NE) does not seem to be intuitive because, if the Entrant does enter, the Incumbent is strictly better off Accommodating

Sequential rationality

- The reason why “incredible threats” are counter-intuitive is because, *conditional on the game reaching a point where the threat must be carried out*, it is not rational to do so
- If players could commit to their strategies (eg by giving a instruction manual to a robot to play in their behalf) then rationality only imposes restrictions ex-ante (at the begining of the game)
- However, if players are not able to commit (eg because they can change the strategy at intermediate stages) then rationality imposes restrictions on behavior *at every point of the game where a decision is made* (ie information sets)
- The notion of sequential rationality generates different refinements of rationalizability and equilibrium, in this class we will only study subgame perfect Nash equilibrium (SPNE)

Subgames

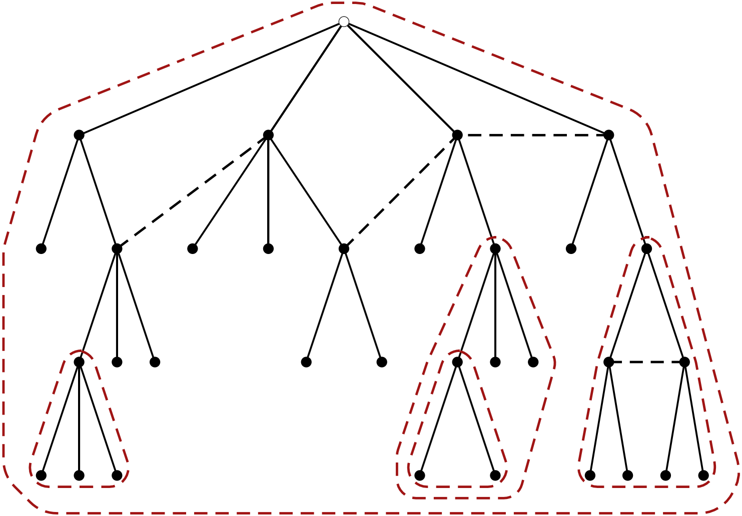
- A subgame is a part of an extensive form game that constitutes a *valid* extensive form game on its own

Definition

A node x initiates a subgame if all the information sets that contain either x or a successor of x contain only nodes that are successors of x. The subgame initialized at x is the extensive form game conformed by x and *all* of its successors

- Notice that the main requirement is that all information sets must be either completely inside or completely outside the subgame
- The whole game is always a subgame, other subgames are called proper subgames
- In a perfect information extensive form game every node initializes a subgame (why?)

Example: Subgames



Subgame perfect equilibrium

Definition

A subgame perfect Nash equilibrium (SPNE) is a strategy profile that induces a Nash equilibrium **on every subgame**

- Since the whole game is always a subgame, every SPNE is a Nash equilibrium, we thus say that SPNE is a *refinement* of Nash equilibrium
- Simultaneous move games have no proper subgames and thus every Nash equilibrium is subgame perfect
- SPNE can be found using a simple algorithm known as backward induction (cf Zermelo 1913)

Example: Entry deterrence with price competition

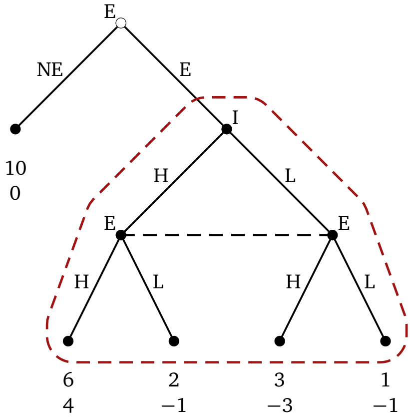
Description

- Consider a market that is originally serviced by a monopolist firm which we call the *incumbent* and a new potential *entrant* is considering to enter the market
- If the entrant stays out of the market, the incumbent can exploit its monopoly power to obtain high profits of $\$10M$
- If the entrant enters the market then they must simultaneously choose between a high price and a low price
- The payoffs resulting from these price choices are given in the following payoff matrix:

		Incumbent	
		H	L
Entrant	H	$4, 6$	$-1, 2$
	L	$-3, 3$	$-1, 1$

Example: Entry deterrence

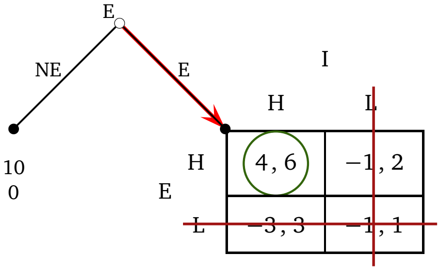
Nash equilibria



	H	L
(E,H)	4 , 6	−1 , 2
(E,L)	−3 , 3	−1 , 1
(NE,H)	0 , 10	0 , 10
(NE,L)	0 , 10	0 , 10

Example: Entry deterrence

Backward induction



- Subgame perfection implies that if the entrant enters then both firms will choose high prices
- Knowing this, the entrant prefers entering to not entering
- $\big((E,H),H\big)$ is the only SPNE

Backward induction

- Backward induction refers to elimination procedures that go as follows:
  1. Identify the “terminal subgames” (ie those without proper subgames)
  2. Pick a Nash equilibrium for each terminal subgame
  3. Replace each terminal subgame with a terminal node where players get the payoffs from the corresponding Nash equilibrium
  4. If there are any non-terminal nodes left go back to step 1

- When there are subgames with multiple equilibria there are different ways of performing backward induction
- For perfect information games this can happen only if there are repeated payoffs

Existence of SPNE

Theorem

For every finite extensive form game, performing backward induction always results in SPNE

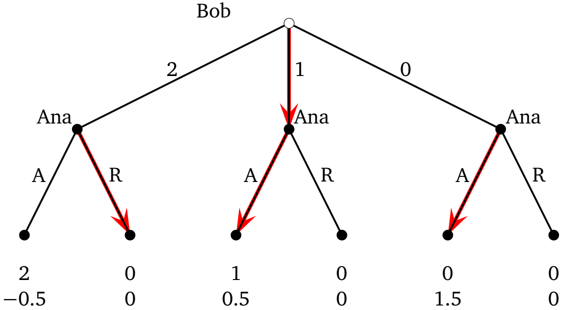
Corollary

Every finite extensive form game has at least one SPNE

Example: Ultimatum Bargaining

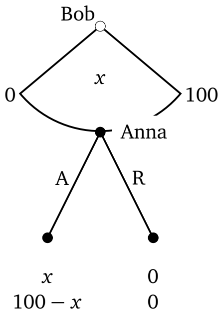
Finite offers

- Anna is thinking about buying a pencil from Bob
- Anna's value for the pencil is \$1.5
- Bob posts a price either \$0, \$1 or \$2 and then Anna decides whether to accept or reject the offer



Example: Ultimatum Bargaining

- Anna and Bob are negotiating on how to split 100$
- Anna makes a take it or leave it offer $(x, 100 - x)$ with $x \in [0, 100]$
- If Bob accepts the offer Anna takes $x$$ and Bob gets the remaining $(100 - x)$$
- If Bob rejects Anna's offer there is no agreement and they both get 0



- In the unique SPNE Bob accepts any $x \leq 100$ and Anna offers $(100, 0)$

Example: Alternate Bargaining

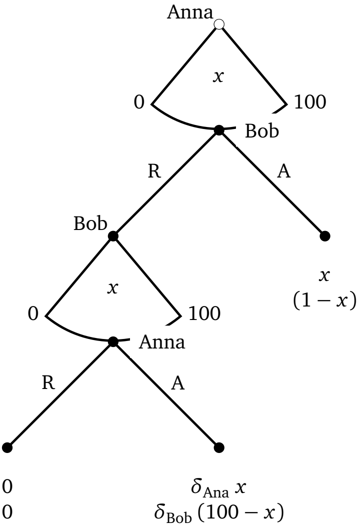
- Now suppose that Anna and Bob take turns in making offers
- In each period the proposer makes an offer $(x, 100 - x)$ and the other player decides whether to accept or to reject
- If an offer is rejected the game goes on to the following round
- Players are impatient and they discount future payoffs with discount rates $\delta_{\text{Ana}}, \delta_{\text{Bob}} \in (0, 1)$
- If the game ends with an offer $(x, 100 - x)$ being accepted at period t, the game ends with payoffs:

$$u_{\text{Ana}} = \delta_{\text{Ana}}^{t} \cdot x$$
$$u_{\text{Bob}} = \delta_{\text{Bob}}^{t} \cdot (100 - x)$$

- If the game ends without agreement both Anna and Bob get 0

Example: Alternate Bargaining

Two periods



Example: Alternate Bargaining

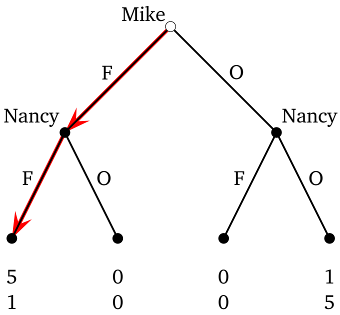
Two periods

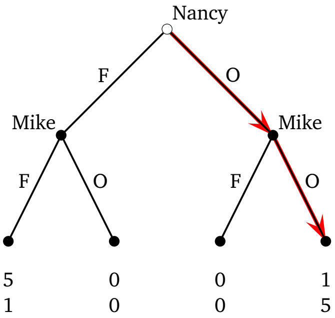
- Suppose that $\delta_{\text{Ana}} = \delta_{\text{Bob}} = \frac{3}{4}$
- *Second period:*
  - On the second period Anna will accept any offer with $x \geq 0$
  - Bob will then offer $(0, 100)$
  - If the game reached the second period Anna would get 0 and Bob would get $\frac{3}{4} \cdot 100 = 75$
- *First period:*
  - On the first period Bob will accept any offer with $100 - x \geq 75$ ie $x \leq 25$
  - Anna will then offer $(25, 75)$
- The game thus will end on the first period with payoffs $(25, 75)$

Example: Sequential Battle of the Sexes

First mover advantage

	Football	Opera
Football	5 , 1	0 , 0
Opera	0 , 0	1 , 5





Example: Stackelberg competition

- Consider a Bertrand duopoly with firms 1 and 2 producing imperfect substitutes with constant marginal cost $c = 5$ and inverse demand functions:

$$D_1(p_1, p_2) = 10 - p_1 + p_2 \qquad D_2(p_1, p_2) = 10 - p_2 + p_1$$

- Assume that choices are not simultaneous:
  - firm 1 is a Stackelberg leader that chooses its price $p_1 \in [0, 20]$ at the beginning of the game
  - firm 2 chooses its price $p_2 \in [0, 20]$ after observing p_1

Example: Stackelberg competition

Backward induction

- Firm 1 know that firm 2 will choose a best response:

$$p_2^* = BR_2(p_1) = 6 + \frac{1}{2}p_1$$

- Hence, firm 1 will choose p_1 to maximize:

$$\begin{aligned} u_1\Big(p_1, BR_2(p_1)\Big) &= (p_1 - 2)\left(10 - p_1 + BR_2(p_1)\right) \\ &= (p_1 - 2)\left(10 - p_1 + \left(6 + \frac{1}{2}p_1\right)\right) \\ &= (p_1 - 2)\left(16 - \frac{1}{2}p_1\right) = -\frac{1}{2}(p_1 - 2)\left(p_1 - 32\right) \end{aligned}$$

- The Stackelberg equilibrium prices are:

$$p_1^S = 17 \qquad p_2^S = 14.5$$

Example: Stackelberg competition

- Payoffs under Stackelberg competition are:

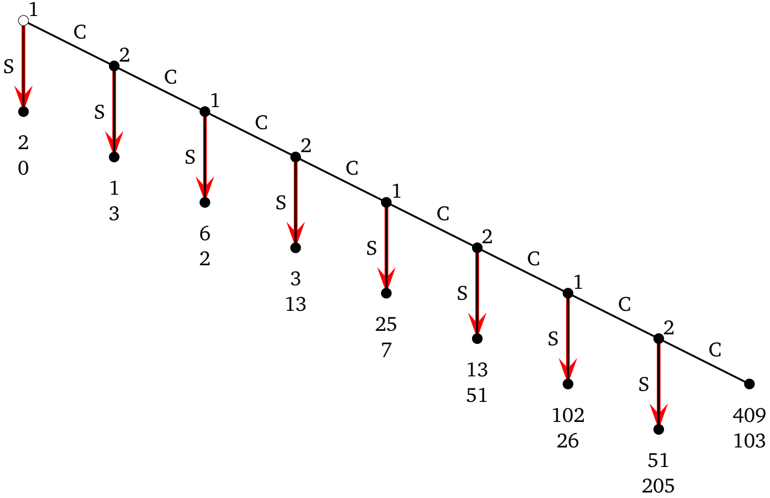
$$u_1(p_1^S, p_2^S) = (17 - 2)(10 - 17 + 14.5) = 112.5$$
$$u_2(p_1^S, p_2^S) = (14.5 - 2)(10 - 14.5 + 17) = 156.25$$

- Under simultaneous Bertrand competition the Nash equilibrium is $(p_1^B, p_2^B) = (12, 12)$ and payoffs are:

$$u_1(p_1^B, p_2^B) = (12 - 2) * (10 - 12 + 12) = 100$$
$$u_2(p_1^B, p_2^B) = (12 - 2) * (10 - 12 + 12) = 100$$

Example: Centipede game

Backward induction can be awkward



Example: Credible government policies

The value of commitment