Econ 4020 – Problem Set II

Due on 03/28

1. Find all the NE, both in pure and mixed strategies, for the following game

	a	b	c	d	e	f
~~v~~	~~0 , 0~~	~~1 , 0~~	~~0 , 0~~	~~5 , 0~~	~~0 , 10~~	~~4 , 0~~
w	1 , 0	<u>4</u> , <u>3</u>	9 , 1	5 , <u>3</u>	0 , 1	2 , 1
~~x~~	~~2 , 2~~	~~0 , 2~~	~~2 , 5~~	~~3 , 2~~	~~8 , 1~~	~~4 , 4~~
y	7 , 3	2 , 4	3 , 0	<u>7</u> , 1	1 , 1	<u>5</u> , <u>6</u>
~~z~~	~~0 , 2~~	~~1 , 3~~	~~8 , 1~~	~~1 , 2~~	~~9 , 1~~	~~0 , 3~~

Iterated dominance — v is dominated by y. Then, e is dominated by b. Then, x is dominated by y. Then, c and a are dominated by b. Then, z is dominated by w. Everything else is rationalizable.

Pure strtegy NE — The only NE in pure strategies are (w,b) and (y,f).

Mixed strategy NE with $\Pr(d) = 0$ — Let $\Pr(b) = q$, $\Pr(f) = 1 - q$, $\Pr(w) = p$ and $\Pr(y) = 1 - p$. Note that

$$U_1(w; q) = 4q + 2(1 - q) = 2 + 2q \quad \text{and} \quad U_1(y; q) = 2q + 5(1 - q) = 5 - 3q$$

and

$$U_2(b; p) = 3p + 4(1 - p) = 4 - p \quad \text{and} \quad U_2(f; p) = 1p + 6(1 - p) = 6 - 5p$$

In order for both w and y to be best responses we need $U_1(w; q) = U_1(y; q)$, that is

$$2 + 2q = 5 - 3q \quad \Leftrightarrow \quad q = \frac{3}{5}$$

In order for both b and f to be best responses we need $U_2(b; p) = U_2(f; p)$, that is

$$4 - p = 6 - 5p \quad \Leftrightarrow \quad p = \frac{1}{2}$$

Hence, the unique mixed NE with $\Pr(d) = 0$ is given by $\sigma_2^*(b) = 3/5$ and $\sigma_1^*(w) = 1/2$.

Mixed strategy NE with $\Pr(d) > 0$ — Note that d is *a* best response, if and only if $\Pr(w) = 1$, in which case b is also a best response and f is *not* a best response. Let $\Pr(b) = q$ and $\Pr(d) = 1 - q$.

$$U_1(w; q) = 4q + 5(1-q) = 5 - q \quad \text{and} \quad U_1(y; q) = 2q + 7(1-q) = 7 - 5q$$

In this case, w is a best response if $U_1(w; q) \geq U_1(y; q)$, that is

$$5 - q \geq 7 - 5q \quad \Leftrightarrow \quad q \geq \frac{1}{2}$$

Hence any strategy profile with $s_1^* = w$, $\sigma_2^*(b) \geq 1/2$ and $\sigma_2^*(d) = 1 - \sigma_2^*(b)$ is a NE.

2. Suppose that Anna and Bob are going to split 100$, and take turns making alternating offers as in the example covered in class. Find the unique SPNE assuming that there are four rounds of Bargaining, and both Anna and Bob have the same discount factor $\delta = 0.5$.

By backward induction:

- On the fourth round, Anna would always accept any positive offer regardless. Bob would then offer $(0, 100)$, and payoffs would be $(0, \delta^3 100) = (0, 12.5)$
- On the third round, Bob would accept a split $(x, 100 - x)$ if and only if $\delta^2(100 - x) \geq \delta^3 100$, that is, if and only if $x \leq 50$. Knowing this, Anna would offer the split $(50, 50)$, and payoffs would be $(\delta^2 50, \delta^2 50) = (12.5, 12.5)$.
- On the second round, Anna would accept a split $(x, 100 - x)$ if and only if $\delta x \geq \delta^2 50$, that is, if and only if $x \geq 25$. Knowing this, Bob would offer the split $(25, 75)$, and payoffs would be $(\delta 25, \delta 75) = (12.5, 37.5)$.
- On the first round, Bob would accept a split $(x, 100 - x)$ if and only if $100 - x \geq \delta 75$, that is, if and only if $x \leq 62.5$. Knowing this, Anna will offer the split $(62.5, 37.5)$, this offer will be accepted, and payoffs will be $(62.5, 37.5)$.

3. Consider the variant of Nim found on transience.com.au/pearl3.html. Use backward induction to find a wining strategy for the first round (with two columns). Each position of the game can be described by (n, m, i) where n is the number of marbles in the shortest row, m is the number of marbles in the longest row, and

i is the player making a move in the present turn. The winning strategy is as follows

- If at the beginning of the game $n \neq m$ choose to start. Otherwise, let the other player start.
- If $n = 0$ and $m > 1$, take $m - 1$ marbles and win the game.
- If $n = 1$ and $m > n$, take m marbles from the longest row and win the game.
- If $n > 2$ and $m > n$, take $m - n$ marbles from the longest row.
- Other cases don't matter because you will never encounter them.

With this strategy, your opponent will always face a board with $n = m$, which means that he will always have to make the rows unequal and you will be back in one of the cases considered before. All this cases either led to another round or end with you winning the game.

4. Suppose that Anna and Bob play the following simultaneous move game twice, and their total payoffs are the sum of the payoffs they get at each stage. Is there a SPNE on which the players choose (A, x) on the first stage? If your answer is positive you must provide the SPNE describing the strategies *in detail*, if it is negative you must explain why this is the case.

	x	y	z
A	7 , 4	1 , 7	2 , 5
B	4 , 5	6 , 6	1 , 2
C	3 , 1	3 , −1	4 , 11
D	8 , 0	6 , 2	0 , 0

Yes, there is. For example, Anna could use the strategy

- On $t = 1$, play A
- On $t = 2$, if the outcome of $t = 1$ was (A, y) or (A, z), play D
- On $t = 2$, if the outcome of $t = 1$ was (D, x), play C
- On $t = 2$, in every other case, play B

Bob could use the strategy

- On $t = 1$, play x
- On $t = 2$, if the outcome of $t = 1$ was (D, x), play z
- On $t = 2$, in every other case, play y

These strategies constitute a SPNE (why?).

5. Four firms produce an homogeneous good in quantities $q_1, q_2, q_3, q_4 \geq 0$, respectively. Each firm has constant marginal cost equal to 10. The market price is given by the inverse demand function

$$P = 100 - q_1 - q_2 - q_3 - q_4$$

(a) Find the NE of the game in which all firms choose quantities independently

Best response functions are given by

$$\text{BR}_i = 45 - \frac{1}{2}\sum_{j \neq i} q_j$$

Knowing that there is a unique equilibrium at it is symmetric, the NE quantity q^C is given by

$$q^C = 45 - \frac{3}{2}q^C \quad \Rightarrow \quad q^C = \frac{90}{5}$$

(b) Find the SPNE of the game in which firm 1 chooses its quantity first, and the remaining firms choose their quantities simultaneously after observing q_1

Taking q_1 ad given, the best response of the other firms is given by

$$\text{BR}_i = 45 - \frac{1}{2}q_1 - \frac{1}{2}\sum_{j \neq i,1} q_j$$

The subgame after which firm 1 chooses q_1 has a unique NE in which all the followers choose the quantity $q^*(q_1)$ given by (why?)

$$q^*(q_1) = 45 - \frac{1}{2}q_1 - q^*(q_1) \quad \Rightarrow \quad q^*(q_1) = \frac{1}{4}(90 - q_1)$$

Anticipating this, the leader will choose q_1 as to maximize

$$q_1\Big(90 - q_1 - 3q^*(q_1)\Big) = \frac{1}{4}q_1(90 - q_1)$$

The optimal quantity for the leader is given by the firs order condition

$$\frac{1}{4}(90 - 2q_1^*) = 0 \quad \Rightarrow \quad q_1^* = 45$$

(c) What is the maximum joint profit that the firms could generate if the quantities they produce were contractible? The joint profits are given by

$$\sum_i q_i \times \left(90 - \sum_i q_i\right)$$

Hence, only the total quantity matters, not who produces it. And the maximum feasible profits are the monopolistic profits generated by a total production of $q^M = 90/2 = 45$. The maximum joint profits are

$$45(90 - 45) = 45^2 = 2025$$

These profits can be generated using a contract, because the minimax for each firm is 0 (why?).

(d) Rank the joint profits of the firms in each of the previous scenarios. The Cournot joint profits are

$$4q^C(90 - 4q^C) = \frac{4}{5}90 \cdot \frac{1}{5}90 = \frac{8100}{25} = 324$$

The total Stackelberg production form part (b) is $q^S = q_1^* + 3q^*(q_1^*) = 90 \cdot 7/8$.

$$q^S(90 - q^S) = \frac{7}{8}90 \cdot \frac{1}{8}90 \approx 885.94$$