Rationality and Dominance

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Reading assignments: Watson, Ch. 4 & 5, and App. B

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uggs vs. rain boots

- Emma would like to wear her Ugg boots today but it might rain
- If it rains, she would prefer to wear her rain boots
- The problem is that she is uncertain about whether it is going to rain
- She believes that it is going to rain with probability $p \in (0, 1)$

	No Rain $[1-p]$	Rain $[p]$
Ugg boots	10	-5
Rain boots	4	6

- Expected utility from wearing her ugg boots

$$U\big(\text{Ugg boots}, p\big) = 10(1-p) - 5p = 10 - 15p$$

- Expected utility from wearing her rain boots

$$U\big(\text{Rain boots}, p\big) = 4(1-p) + 6p = 4 + 2p$$

- Emma will choose to wear her ugg boots if and only if

$$U\big(\text{Ugg boots}, p\big) \geq U\big(\text{Rain boots}, p\big) \quad \Leftrightarrow \quad p \leq \frac{6}{17} \approx 35\%$$

- Uncertainty $\approx$ lack of information
- A player is uncertain about an event if he does not know whether the event holds or not
- Beliefs are probability functions representing likelihood assessments
- Maintained assumption:

> Players make choices to maximize their expected utility given their beliefs

- Flip a fair coin until it lands tails
- If we flipped the coin n times, you get \$$2^n$
- How much would you be willing to pay to participate?

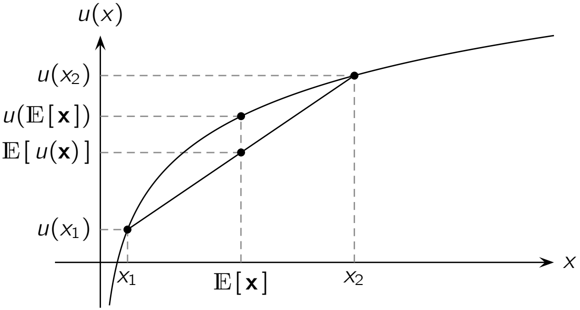
$$\mathbb{E}[\,2^{\mathrm{n}}\,] = \frac{1}{2}\cdot 2 + \frac{1}{4}\cdot 4 + \frac{1}{8}\cdot 8 + \ldots = \sum_{n=1}^{\infty}\frac{1}{2^n}\cdot 2^n = \infty$$

$$\mathbb{E}[\,\log(2^{\mathrm{n}})\,] = \sum_{n=1}^{\infty}\frac{1}{2^n}\cdot\log\left(2^n\right) = \log(2)\sum_{n=1}^{\infty}\frac{1}{2^n}\cdot n = 2\log(2) \approx 0.60$$

- When it comes down to monetary prizes
  - Risk neutrality – maximize expected value
  - Risk aversion – maximize the expectation of a concave utility function
  - An agent is risk averse if and only if

$$\mathbb{E}[\, u(\mathbf{x}) \,] \leq u(\mathbb{E}[\, \mathbf{x} \,])$$

for every random variable $\mathbf{x}$ (Jensen's inequality)



beliefs

- Consider a strategic form game with independent choices
- Each player might be uncertain about his opponents' strategies

> Given a strategic form game, a belief for player $i \in I$ is a probability distribution θ_{-i} over his opponent's strategies

- $\theta_{-i}(s_{-i})$ is the likelihood that i assigns to his opponents' choosing s_{-i}
- If S_{-i} has N elements, then a belief for i is a vector consisting of N numbers between 0 and 1 that add up to 1
- If S_{-i} has two elements, then a belief for i can be characterized by a single number $p \in [0, 1]$

	Football [p]	Opera [$1-p$]
Football	5 , 1	0 , 0
Opera	0 , 0	1 , 5

- A belief for Mike consists of two numbers $\theta_N(F)$ and $\theta_N(O)$ between 0 and 1 such that $\theta_N(F) + \theta_N(O) = 1$
- Simpler notation $p = \theta_N(F)$ and $(1-p) = \theta_N(O)$
- p is the probability that Mike assigns to Nancy going to the football game and $(1-p)$ is the probability that Mike assigns to Nancy going to the Opera

- Fix i's beliefs θ_{-i} about his opponents' behavior
- The expected payoff or expected utility for i from choosing s_i is

$$U_i(s_i, \theta_i) = \mathbb{E}_{\theta_i}\left[\, u_i(s_i, \mathbf{s}_{-i})\,\right]$$

- For finite games, expected utility is jut a weighted sum if payoffs weighted by their likelihoods

$$U_i(s_i, \theta_i) = \sum_{s_{-i} \in S_{-i}} \theta_{-i}(s_{-i}) u_i(s_i, s_{-i})$$

	Football $[p]$	Opera $[1-p]$
Football	5 , 1	0 , 0
Opera	0 , 0	1 , 5

- Given his beliefs, Mike's expected utility for going to the football game is:

$$U_M(\text{Football}, p) = 5 \cdot p + 0 \cdot (1-p) = 5p$$

- His s expected utility for going to the opera is:

$$U_M(\text{Opera}, p) = 0 \cdot p + 1 \cdot (1-p) = 1-p$$

battle of the sexes

	Football [p]	Opera [$1-p$]
Football [q]	5 , 1	0 , 0
Opera [$1-q$]	0 , 0	1 , 5

- Given her beliefs, Nancy's expected utility for going to the football game is:

$$U_N(\text{Football}, q) = 1 \cdot q + 0 \cdot (1-q) = q$$

- His s expected utility for going to the opera is:

$$U_N(\text{Opera}, q) = 0 \cdot q + 5 \cdot (1-q) = 5 - 5q$$

example – 4×4 game

	A $[\theta_2(A)]$	B $[\theta_2(B)]$	C $[\theta_2(C)]$	D $[\theta_2(D)]$
a $[\theta_2(a)]$	7 , 9	4 , 5	6 , 4	2 , 2
b $[\theta_2(b)]$	2 , 5	5 , 2	8 , 6	9 , 8
c $[\theta_2(c)]$	5 , 4	2 , 1	1 , 3	4 , 5
d $[\theta_2(d)]$	1 , 8	4 , 7	4 , 4	1 , 9

$$U_1(a, \theta_2) = 7\theta_2(A) + 4\theta_2(B) + 6\theta_2(C) + 2\theta_2(D)$$

$$U_1(c, \theta_2) = 5\theta_2(A) + 2\theta_2(B) + \theta_2(C) + 4\theta_2(D)$$

$$U_2(B, \theta_1) = 5\theta_1(a) + 2\theta_1(b) + \theta_1(c) + 7\theta_1(d)$$

$$U_2(D, \theta_1) = 2\theta_1(a) + 8\theta_1(b) + 5\theta_1(c) + 9\theta_1(d)$$

uneven thumbs

Up $[\theta_3(\text{Up})]$

	Up $[\theta_2(\text{Up})]$	Down $[\theta_2(\text{Up})]$
Up	0 , 0 , 0	1 , −1 , 1
Down	−1 , 1 , 1	1 , 1 , −1

Down $[\theta_3(\text{Down})]$

	Up $[\theta_2(\text{Up})]$	Down $[\theta_2(\text{Up})]$
Up	1 , 1 , −1	−1 , 1 , 1
Down	1 , −1 , 1	0 , 0 , 0

$$U_1(\text{Up}, \theta_{-1}) = \theta_2(\text{Up})\theta_3(\text{Down}) + \theta_2(\text{Down})\theta_3(\text{Up}) - \theta_2(\text{Down})\theta_3(\text{Down})$$

$$U_1(\text{Down}, \theta_{-1}) = \theta_2(\text{Up})\theta_3(\text{Down}) + \theta_2(\text{Down})\theta_3(\text{Up}) - \theta_2(\text{Up})\theta_3(\text{Up})$$

- Firms $\{1, 2\}$ choose prices $p, q \in [0, 10]$ and make profits

$$u_1(p,q) = -p^2 + \left(12 + \frac{1}{2}q\right)p - \left(20 + q\right)$$
$$u_2(p,q) = -q^2 + \left(12 + \frac{1}{2}p\right)q - \left(20 + p\right)$$

- Firm 1's expected utility is given by:

$$\begin{aligned} U_1(p,\theta_2) &= \mathbb{E}_{\theta_2}\left[-p^2 + \left(12 + \frac{1}{2}\mathbf{q}\right)p - \left(20 + \mathbf{q}\right)\right] \\ &= -p^2 + \left(12 + \frac{1}{2}\bar{q}\right)p - \left(20 + \bar{q}\right) \end{aligned}$$

where $\bar{q} = \mathbb{E}_{\theta_2}[\,\mathbf{q}\,]$

A strategy $s_i \in S_i$ is a best response to a belief θ_{-i} if and only if it maximizes $U_i(\,\cdot\,, \theta_{-i})$, i.e., if and only if

$$U_i(s_i, \theta_{-i}) \geq U_i(s_i', \theta_{-i})$$

for every other strategy $s_i' \in S_i$

- $\mathrm{BR}_i(\theta_{-i}) \subseteq S_i$ denotes the set of i's best responses to θ_i
- Rational agents choose strategies in $\mathrm{BR}_i(\theta_{-i})$

- Mike's expected utility functions in the Battle of the Sexes

$$U_M(\text{Football}, p) = 5p \qquad U_M(\text{Opera}, p) = 1 - p$$

- Going to the football game is a best response if and only if

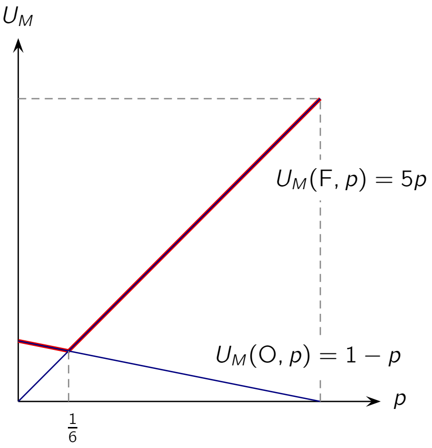
$$U_M(\text{Football}, p) \geq U_M(\text{Opera}, p) \quad \Leftrightarrow \quad p \geq \frac{1}{6}$$

- Going to the opera game is a best response if and only if

$$U_M(\text{Football}, p) \leq U_M(\text{Opera}, p) \quad \Leftrightarrow \quad p \leq \frac{1}{6}$$

- Mike is indifferent when $p = \frac{1}{6}$

battle of the sexes



- Derivative $\sim$ slope: positive if increasing, negative if decreasing
- Second derivative $\sim$ curvature: negative if concave
- Derivatives of polynomials

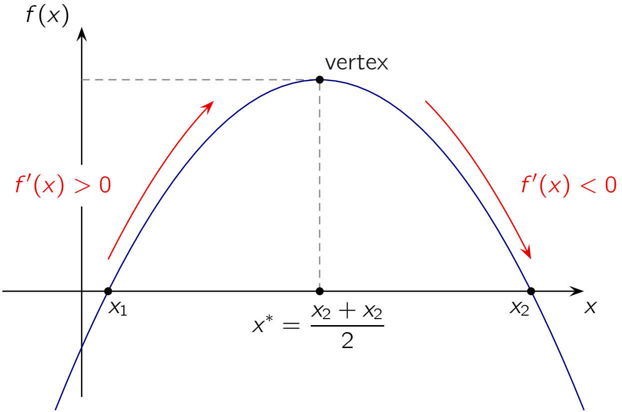
$$f(x) = x^r \quad \Rightarrow \quad f'(x) = rx^{r-1}$$

$$f(x) = a \cdot g(x) + h(x) \quad \Rightarrow \quad f'(x) = a \cdot g'(x) + h'(x)$$

$$f(x) = g(x)h(x) \quad \Rightarrow \quad f'(x) = h(x)g'(x) + g(x)h'(x)$$

Any concave differentiable function f is maximized at points that satisfy the first order condition $f'(x) = 0$

quadratic functions



$$f(x) = -(x - x_1)(x - x_2) = -x^2 + (x_1 + x_2)x - x_1 x_2$$

$$f'(x) = -2x + (x_1 + x_2)$$

- Firm 1's expected utility

$$U_1(p, \theta_1) = -p^2 + \left(12 + \frac{1}{2}\bar{q}\right) p - \left(20 + \bar{q}\right)$$

- Think of U_1 as a function of p taking θ_1 as a parameter

$$U_1'(p) = -2p + \left(12 + \frac{1}{2}\bar{q}\right)$$

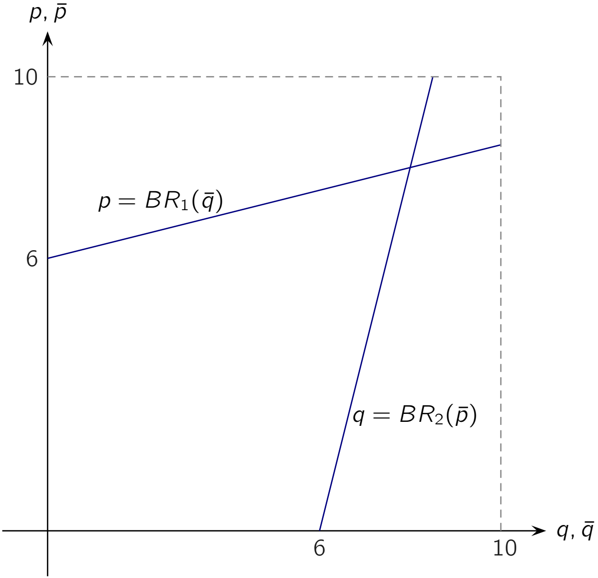
- The first order condition is

$$-2p + \left(12 + \frac{1}{2}\bar{q}\right) = 0$$

- It has a unique best response

$$p = 6 + \frac{1}{4}\bar{q}$$

bertrand competition



- Rational players choose best response to their beliefs
- What predictions can we make if we don't know their beliefs?

> Rational players can only choose a strategy if it is a best response to some belief

- The set of (first order) rational strategies for player i is

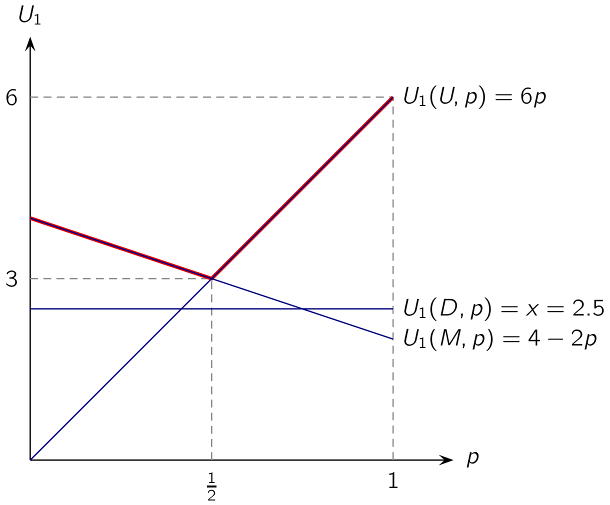
$$\mathrm{B}_i = \left\{ s_i \in S_i \;\middle|\; \text{there is some } \theta_{-i} \text{ such that } s_i \in \mathrm{BR}_i(\theta_{-i}) \right\}$$

	L p	R $[1-p]$
U	6 , 3	0 , 1
M	2 , 1	4 , 0
D	x , 2	x , 1

- Player 1's expected utility is given by:

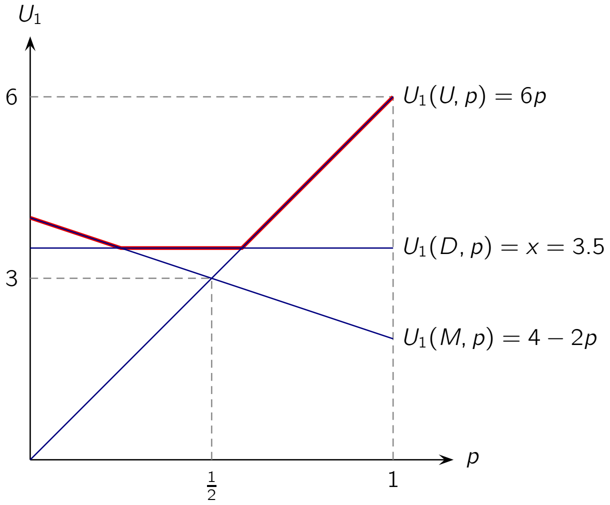
$$U_1(U, p) = 6p \qquad U_1(M, p) = 4 - 2p \qquad U_1(D, p) = x$$

example – 3×2 game



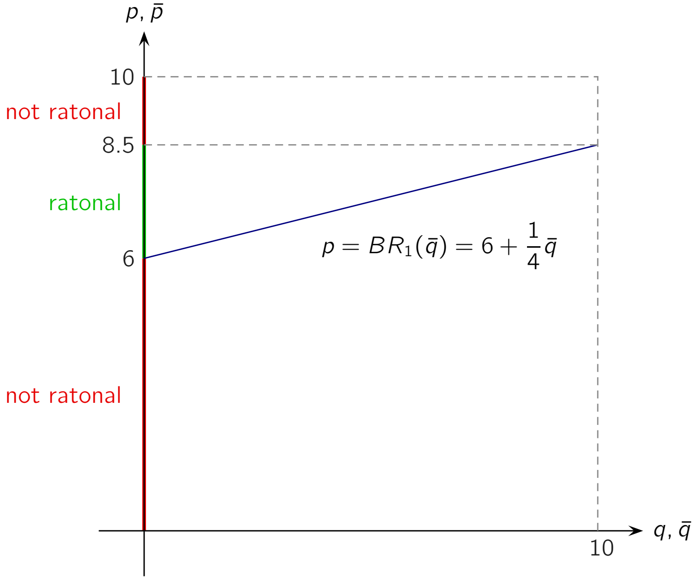
If $x < 3$, then D is never a best response

example – 3×2 game



If $x > 3$, then D is a best response to $p = 1/2$

bertrand competition



strictly dominance

- Finding the set of best responses is not always straightforward
- Easier to work with strictly dominated strategies
- Strict dominance is as an interesting concept on its own
- We care about its relation with rationality — a strategy is rational if and only if it is not strictly dominated

During WW2, Arrow was assigned to a team of statisticians to produce long-range weather forecasts. After a time, Arrow and his team determined that their forecasts were not much better than pulling predictions out of a hat. They wrote their superiors, asking to be relieved of the duty. They received the following reply, and I quote "The Commanding General is well aware that the forecasts are no good. However, he needs them for planning purposes".

— David Stockton, FOMC Minutes, 2005

- Allow players to randomize their choices

> A mixed strategy for player i is a probability distribution σ_i over his strategies

- Mathematically, beliefs and mixed strategies are similar but the interpretation is different
- For example, in a game with two players 1 and 2
  - θ_2 represents 1's beliefs about 2's behavior, which might be deterministic
  - σ_2 represents 2's behavior, which could be unknown by 1

- i's expected utility for playing according to σ_i

$$U_i(\sigma_i, s_{-i}) = \mathbb{E}_{\sigma_i}\left[\, u_i(\mathbf{s}_i, s_{-i})\,\right]$$

A pure strategy s_i is strictly dominated by a pure or mixed strategy σ_i if playing according σ_i gives i a strictly higher expected utility regardless of what other players do, i.e., if

$$U_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i}) \qquad \text{for every } s_{-i} \in S_{-i}$$

- Let UD_i denote the set of undominated strategies for i

example – 3×2 game

	L	R
U	6 , 3	0 , 1
M	2 , 1	4 , 0
D	2.5 , 2	2.5 , 1

- For player 2, R is strictly dominated by L because

$$u_2(U, L) = 3 > 1 = u_2(U, R)$$

$$u_2(M, L) = 1 > 0 = u_2(M, R)$$

$$u_2(D, L) = 2 > 1 = u_2(D, R)$$

example – 3×2 game

	L	R
U	6 , 3	0 , 1
M	2 , 1	4 , 0
D	2.5 , 2	2.5 , 1

- For player 1, D is not strictly dominated U nor by M
- It is strictly dominated by $\sigma_1 = (1/3, 2/3, 0)$ because

$$U_1(\sigma_1, L) = \frac{1}{3}6 + \frac{2}{3}2 = \frac{10}{3} > 2.5 = u_1(D, L)$$

$$U_1(\sigma_1, R) = \frac{2}{3}4 = \frac{8}{3} > 2.5 = u_1(D, R)$$

A strategy s_i is rational if and only if it is not dominated by any other pure or mixed strategy, i.e., $\mathrm{UD}_i = \mathrm{B}_i$

- Rational players always choose best responses
- We can find rational actions by eliminating strictly dominated strategies
- In many cases it suffices to consider dominance *by pure strategies*
- Finding all actions that are dominance by pure or mixed strategies is computationally similar to finding convex hulls

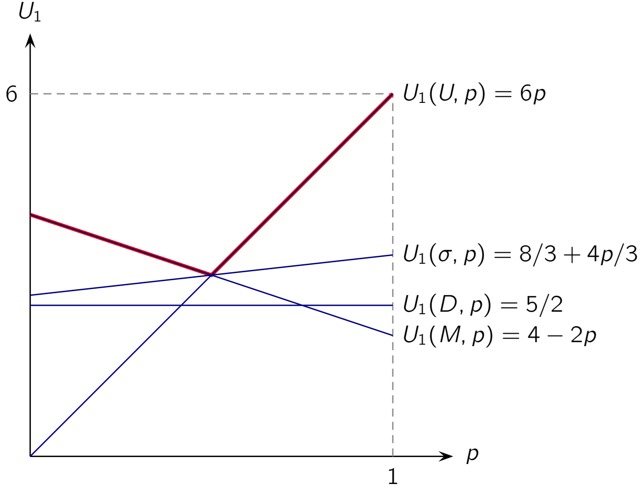
$\mathrm{B}_i \subseteq \mathrm{UD}_i$ in finite games

- Suppose the game is finite and take a rational action s_i^0
- s_i^0 is a best response to some belief θ_{-i}
- Suppose towards a contradiction that s_i^0 is dominated by some σ_i, then

$$\begin{aligned} U_i(s_i^0, \theta_i) &= \sum_{s_{-i}} \theta_{-i}(s_{-i}) \cdot u_i(s_i^0, s_{-i}) < \sum_{s_{-i}} \theta_{-i}(s_{-i}) \cdot U_i(\sigma_i, s_{-i}) \\ &= \sum_{s_{-i}} \sum_{s_i} \theta_{-i}(s_{-i}) \cdot \sigma_i(s_i) \cdot u_i(s_i, s_{-i}) \\ &= \sum_{s_i} \sigma_i(s_i) \cdot \left(\sum_{s_{-i}} \theta_{-i}(s_{-i}) \cdot u_i(s_i, s_{-i}) \right) = \sum_{s_i} \sigma_i(s_i) \cdot U_i(s_i, \theta_{-i}) \end{aligned}$$

- This would imply that $U_i(s_i^0, \theta_{-i}) < U_i(s_i, \theta_{-i})$ for some $s_i \in S_i$ ▼
- Hence, s_i^0 is undominated

$\mathrm{UD}_i \subseteq \mathrm{B}_i$ **in** 3×2 **example**



If $x = 5/2$, then D is never a best response

and it is dominated by $\sigma_1 = (2/3, 1/3, 0)$

- In some few cases, eliminating dominated strategies is sufficient to determine a unique outcome

	Keep Silent	Confess
Keep silent	-1 , -1	-5 , 0
Confess	0 , -5	-3 , -3

- In the prisoner's dilemma, keeping silent is strictly dominated by confessing
- Therefore, rational players playing the prisoner's dilemma will confess
- When is this a good prediction?

- Anna and Bob work as partners
- Each provides effort in $[0, 20]$
- Let A and B denote the levels of effort provided by Anna Bob
- Effort has a cost of $-A^2$ for Anna and $-B^2$ for Bob
- The firm's revenues are given by

$$R(A, B) = 4A + 2B$$

- Anna and Bob split the firm's revenues evenly so that payoffs are

$$u_{\text{Anna}}(A, B) = 2A + B - \frac{1}{2}A^2$$
$$u_{\text{Bob}}(A, B) = 2A + B - \frac{1}{2}B^2$$

- Anna's expected utility is given by

$$U_{\text{Anna}}(A, \theta_{\text{Bob}}) = 2A + \mathbb{E}_{\theta_{\text{Bob}}}[\,\mathbf{B}\,] - \frac{1}{2}A^2$$

- Therefore

$$U'_{\text{Anna}}(A) = 2 - A \qquad \& \qquad U''_{\text{Anna}}(A) = -1$$

- Hence, U'_{Anna} is strictly concave
- Anna's best response is given by the first order condition

$$U'_{\text{Anna}}(A^*) = 0 \qquad \Leftrightarrow \qquad A^* = 2$$

- Since A^* maximizes Anna's expected utility regardless of her beliefs, every other level of effort is strictly dominated

- Some people like to distinguish between rationalizability and correlated rationalizability
- For more than two players, the original definition of rationalizability required independent beliefs, i.e., $\theta_{-i} = \prod_{j \neq i} \theta_j$
- If we imposed this requirement, we could have $\mathrm{B}_i \subsetneq \mathrm{UD}_i$
- When does this requirement make sense?

	L	R
T	1 , 0	5 , 1
B	1 , 1	1 , 0

- Would you ever consider playing B?
- Not if you were rational and assigned any positive probability to R (cautiousness?)
- A form of weak dominance will become important when we go back to extensive form games