Equilibrium

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Reading assignments: Watson, Ch. 9, 10 & 11

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pure strategy nash equilibrium

rationalizability vs. equilibrium

- Strength of rationality/rationalziability
  - Strong ties to decision theory
  - Relatively weak assumptions (?)
- Drawbacks rationality/rationalziability
  - Weak predictions
  - Specially with low levels of sophistication
  - Allows for “erroneous” beliefs
- An alternative is to assume that players beliefs are correct
- Resulting solution concepts are called equilibria

- Suppose the players discuss and agree on some strategy profile $s = (s_1, \ldots, s_n)$ before playing the game
- After that, players go different ways and choose strategies independently
- Suppose player i believes that his/her opponents will not deviate from the intended strategy profile
- Then, i wants to choose s_i if and only if it is a best response to s_{-i}
- That is, if and only if, i can not strictly benefit from unilaterally deviating from the intended strategy profile
- If no players have strict incentives to deviate unilaterally then the plan is self-enforceable, and we call it a Nash equilibrium

A Nash equilibrium in pure strategies (PNE) is a strategy profile $s \in S$ such that no player can *strictly* gain from *unilaterally* deviating, i.e.,

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

for every player i and every alternative strategy $s_i' \in S_i$

- Equivalently, a PNE is a profile of strategies $s \in S$ which are best responses to each other, i.e., such that $s_i \in BR_i(s_{-i})$ for every player i
- In a bimatrix game, a pair of strategies is a PNE if player 1 is maximizing his payoff along the *column*, and player 2 is maximizing her payoff along the *row*

example – a 4×4 game

	a	b	c	~~d~~
w	$0, \underline{7}$	$2, 5$	$\underline{7}, 0$	~~$0, 1$~~
x	$5, 2$	$\underline{3}, \underline{3}$	$5, 2$	~~$0, 1$~~
y	$\underline{7}, 0$	$2, 5$	$0, \underline{7}$	~~$0, 1$~~
~~z~~	~~$0, 0$~~	~~$0, -2$~~	~~$0, 0$~~	~~$10, -1$~~

example – battle of the sexes

	Football	Opera
Football	5 , 1	0 , 0
Opera	0 , 0	1 , 5

- To find PNE a matrix game, one can start by highlighting the best response payoffs for each player
- Cells with all payoffs highlighted correspond to PNE
- Are these good predictions? When?

assumptions

- Rationalizability
  - Rationality
  - Common knowledge of rationality
- Equilibrium in pure strategies
  - Rationality
  - Deterministic choices
  - Correct beliefs
- Brandenburger (1992) *Knowledge and Equilibrium in Games*. Journal of Economic Perspectives

why correct beliefs?

- *Communication* – If players communicate prior to playing the game, they might agree to play certain way
- *Institutions* – Institutions/mediators might help to coordinate players expectations
- *Learning* – If players interact repeatedly they might learn from experience how to predict their opponents behavior
- *Dynamic heuristics* – Simple adaptive rules (e.g. do things that you regret not having done in the past) can converge to equilibria
- *Imitation/selection* – Dynamics resulting from the persistence of successful behavior via selection or adaptation (memes) might converge to equilibrium
- Focal points – Some strategies might naturally draw the attention of the players

Proposition — PNE strategies are rationalizable

Proof:

- Suppose s^* is a PNE
- Best responses are undominated
- As long as s^*_{-i} has not been eliminated, s^*_i cannot be eliminated
- Hence, s^* survives iterated dominance

Proposition — In finite games, if there is a unique rationalizable strategy profile, then it is a PNE

Proof:

- Suppose s^0 is rationalizable, and thus never eliminated
- If s_i' is a best response to s_{-i}^0 of i, it would never be eliminated
- Since there is a unique rationalizable strategy for each player, $s_i' = s_i^0$
- Hence, s_i^0 is a best response to s_{-i}^0

classic 2×2 examples

	Full	Empty
~~Full~~	~~3 , 3~~	~~0 , 5~~
Empty	5 , 0	2 , 2

	Continue	Swerve
Continue	0 , 0	5 , 1
Swerve	1 , 5	2 , 2

	GCS	PS
GCS	1 , 1	0 , 0
PS	0 , 0	2 , 2

	Press	Don't press
Press	3 , 1	0 , 5
Don't press	6 , -2	~~1 , 1~~

- Firms 1 and 2 choosing quantities $q_1, q_2 \geq 0$
- Constant marginal costs $c = 10$ and inverse demand function

$$P(q_1, q_2) = 100 - q_1 - q_2$$

- Profit functions (payoffs)

$$u_1(q_1, q_2) = (90 - q_2 - q_1)q_1 \qquad u_2(q_1, q_2) = (90 - q_1 - q_2)q_2$$

- Best responses to pure strategies

$$BR_1(q_2) = 45 - \frac{1}{2}q_2 \qquad BR_2(q_1) = 45 - \frac{1}{2}q_1$$

- A PNE is a pair $q_1^*, q_2^* \geq 0$ of mutual best responses

$$q_1^* = \mathrm{BR}_1(q_2^*) \qquad q_2^* = \mathrm{BR}_2(q_1^*)$$

- Using our formula for best responses

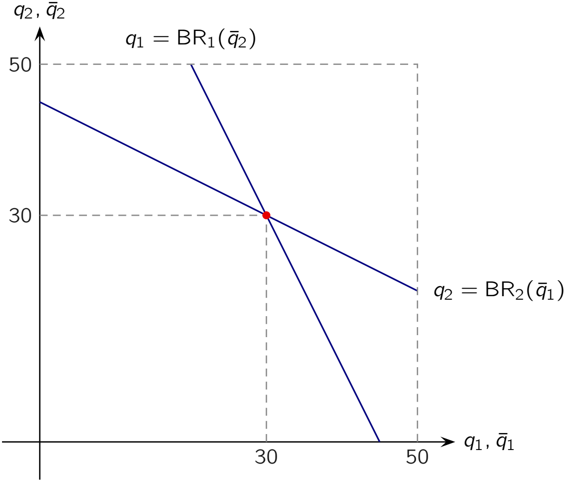
$$q_1^* = 45 - \frac{1}{2}q_2^* \quad \text{and} \quad q_2^* = 45 - \frac{1}{2}q_1^*$$

$$\Rightarrow \quad q_2^* = 45 - \frac{1}{2}\left(45 - \frac{1}{2}q_2^*\right) = \frac{1}{2}45 + \frac{1}{4}q_2^*$$

$$\Rightarrow \quad 3q_2^* = 90 \quad \Rightarrow \quad q_2^* = 30$$

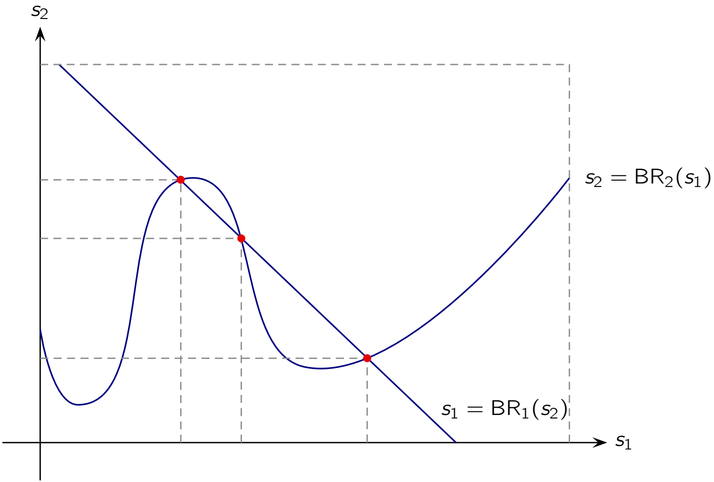
$$\Rightarrow \quad q_1^* = 45 - \frac{1}{2}30 = 45 - 15 = 30$$

- So the game has a unique PNE $(30, 30)$
- Recall that this was the *unique* rationalizable strategy profile



The NE is given by the intersection of BR curves

example – multiple NE



location game

	1	2	3	4	5	6	7
1	35, 35	10, <u>60</u>	15, 55	20, 50	25, 45	30, 40	35, 35
2	<u>60</u>, 10	35, 35	20, <u>50</u>	25, 45	30, 40	35, 35	40, 30
3	55, 15	<u>50</u>, 20	35, 35	30, <u>40</u>	35, 35	40, 30	45, 25
4	50, 20	45, 25	<u>40</u>, 30	<u>35</u>, <u>35</u>	<u>40</u>, 30	45, 25	50, 20
5	45, 25	40, 30	35, 35	30, <u>40</u>	35, 35	<u>50</u>, 20	55, 15
6	40, 30	35, 35	30, 40	25, 45	20, <u>50</u>	35, 35	<u>60</u>, 10
7	35, 35	30, 40	25, 45	20, 50	15, 55	10, <u>60</u>	35, 35

	Rock	Paper	Scissors
Rock	$0\ ,\ 0$	$-1\ ,\ \underline{1}$	$\underline{1}\ ,\ -1$
Paper	$\underline{1}\ ,\ -1$	$0\ ,\ 0$	$-1\ ,\ \underline{1}$
Scissors	$-1\ ,\ \underline{1}$	$\underline{1}\ ,\ -1$	$0\ ,\ 0$

youtube.com/watch?v=fVH7dxyr3Qc

Batzilis, Jaffe, Levitt, List & Picel (2016) *mimeo*

equilibrium with mixed strategies

During WW2, Arrow was assigned to a team of statisticians to produce long-range weather forecasts. After a time, Arrow and his team determined that their forecasts were not much better than pulling predictions out of a hat. They wrote their superiors, asking to be relieved of the duty. They received the following reply, and I quote "The Commanding General is well aware that the forecasts are no good. However, he needs them for planning purposes".

— David Stockton, FOMC Minutes, 2005

- In strictly competitive situations, players might want to remain unpredictable
- One way to do so is by using mixed strategies is by randomizing

> A mixed strategy for player i is a probability distribution σ_i over his strategies

- Randomization can take different forms
  - Rolling a dice
  - Conditioning on random events or feelings
  - Complex patterns

- i's expected utility for playing given mixed strategies $\sigma = (\sigma_i, \sigma_{-i})$

$$\begin{aligned} U_i(\sigma) &= \mathbb{E}_\sigma \left[u_i(\mathbf{s}_i, \mathbf{s}_{-i}) \right] \\ &= \sum_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} \sigma_i(s_i)\sigma_{-i}(s_{-i})u_i(s_i, s_{-i}) \quad \text{(for finite games)} \end{aligned}$$

A Nash equilibrium (NE) is a profile of pure or mixed strategies σ such that no player can *strictly* gain from *unilaterally* deviating, i.e.,

$$U_i(\sigma_i, \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i})$$

for every player i and every alternative strategy $\sigma_i' \in \Delta(S_i)$

	Rock	Paper	Scissors
Rock	0 , 0	-1 , 1	1 , -1
Paper	1 , -1	0 , 0	-1 , 1
Scissors	-1 , 1	1 , -1	0 , 0

- Suppose the row player randomizes uniformly
- Then, player 2's expected payoff is *for any strategy* is 0
- Hence, both players choosing $\sigma_i = (1/3,\ 1/3,\ 1/2)$ is a NE

alternative interpretations

- Do players really randomize? maybe (Arrow's anecdote)
- A mixed strategy NE could represent things other than randomization
  - Subjective beliefs
  - Proportions in a large population
  - Frequencies over time

Proposition — If a rational player randomizes, she must be indifferent between all the strategies she chooses with positive probability

Proof:

- Suppose $u_i(s_i, \theta_{-i}) < u_i(s_i', \theta_{-i})$
- Suppose σ_i assigns positive probability to both s_i and s_i'
- Let σ_i' be as σ_i, except that all the probability that σ_i assigns to s_i, σ_i' assigns it to s_i'
- It is easy to verify that $U_i(\sigma_i', \theta_{-i}) > U_i(\sigma_i, \theta_{-i})$

- The previous proposition asserts that players who randomize must be indifferent between all the strategies with positive probability
- This fact helps to find mixed strategy NE
  1. "Guess" which strategies are in the support of the mixtures
     - Be smart, e.g., ignore dominated strategies
  2. For each player i, look for a mixed strategy for $-i$ that makes i be indifferent between these strategies

	L [p]	R [$1-p$]
U [q]	3 , 3	5 , 8
D [$1-q$]	1 , 2	6 , 1

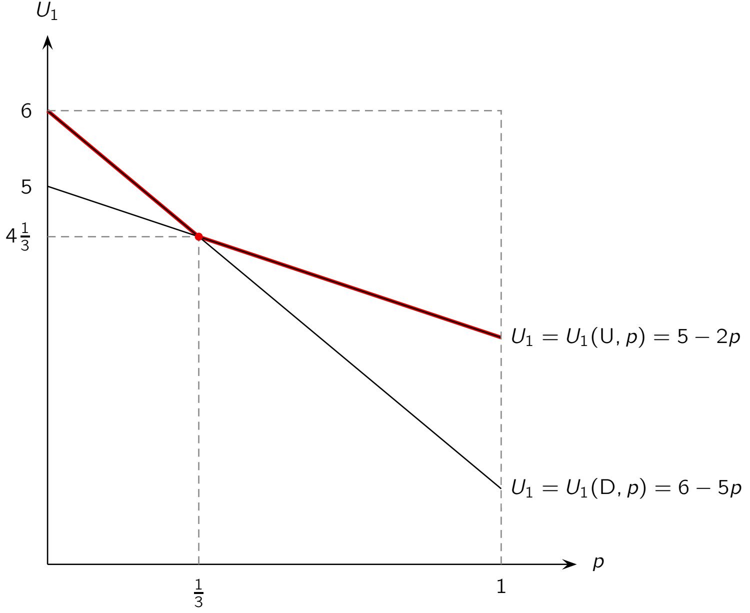
- Row's expected utility for each pure strategy is

$$U_1(U, p) = 3p + 5(1-p) = 5 - 2p$$
$$U_1(D, p) = 1p + 6(1-p) = 6 - 5p$$

- Row is indifferent between U and D if $U_1(U, p) = U_1(D, p)$

$$5 - 2p = 6 - 5p \quad \Leftrightarrow \quad p = \frac{1}{3}$$

example – 2×2 game



example – 2×2 game

	L $[p]$	R $[1-p]$
U $[q]$	3 , 3	5 , 8
D $[1-q]$	1 , 2	6 , 1

- Col's expected utility for each pure strategy is:

$$U_2(L,q) = 3q + 2(1-q) = 2 - q$$
$$U_2(R,q) = 8q + 1(1-q) = 7q - 1$$

- Col is thus indifferent between L and R if and only if $U_2(L,q) = U_2(R,q)$

$$2 - q = 7q - 1 \quad \Leftrightarrow \quad q = \frac{1}{6}$$

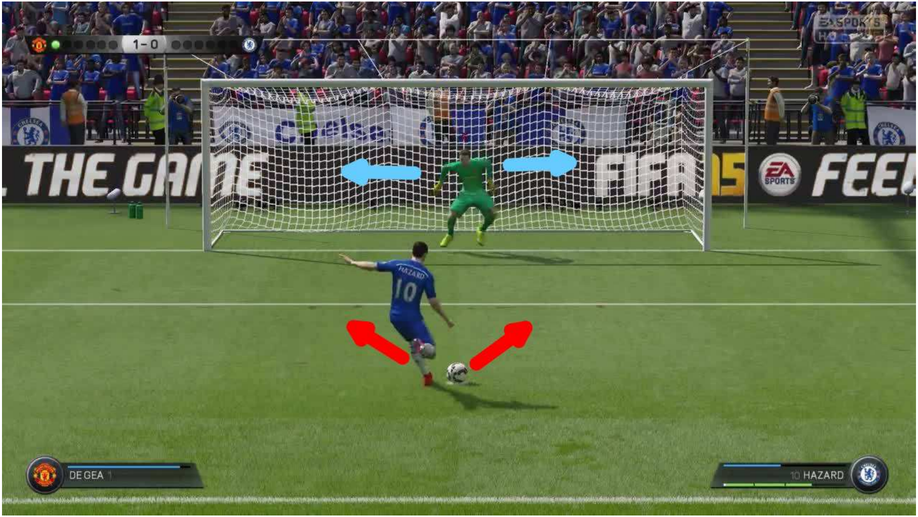
		Col	
		L $[p]$	R $[1-p]$
Row	U $[q]$	3 , 3	5 , 8
	D $[1-q]$	1 , 2	6 , 1

- We then have found a mixed equilibrium in pure strategies:

$$\sigma_1 = \left(\frac{1}{6}, \frac{5}{6}\right)$$
$$\sigma_2 = \left(\frac{1}{3}, \frac{2}{3}\right)$$

- A player randomizing in a NE must be indifferent
- Indifference is a consequence of equilibrium conditions, not an assumption

- Why bother making my opponent indifferent?
  - Purification results
  - Equilibrium of dynamic process
  - Empirical support (in some cases)


1-0
THE GAME
FIFA 15
EA SPORTS
FEEL
HAZARD
10
DE GEA 1
10 HAZARD

- Chiappori, Levitt & Groseclose (2002)
  Testing Mixed-Strategy Equilibria When Players Are Heterogeneous

- Shooter wants to maximize the probability of scoring
- Keeper wants to minimize the probability of scoring
- Unique equilibrium in mixed strategies

- Probability of scoring should not depend on the direction of the kick, adjusting for heterogeneity

- Look at 500 penalty kicks from professional European League games
- Cannot reject the hypothesis of equal scoring probabilities

- Gaurioty, Pagez & Wooders (2016)
  Nash at Wimbledon: Evidence from Half a Million Serves

Example: A 4×4 game

	~~a~~	b	c	~~d~~
~~w~~	~~0 , 9~~	~~0 , 4~~	~~0 , 2~~	~~0 , 6~~
x	~~2 , 1~~	9 , 3	1 , 7	~~2 , 2~~
~~y~~	~~7 , 1~~	~~0 , 0~~	~~3 , 5~~	~~0 , 2~~
z	~~2 , 1~~	1 , 8	4 , 0	~~1 , 4~~

- Let p be the probability of b and $1 - p$ the probability of c, for indifference we must have:

$$9p + (1 - p) = p + 4(1 - p) \quad \Leftrightarrow \quad p = \frac{3}{11}$$

- Let q be the probability of x and $1 - q$ the probability of z, for indifference we must have:

$$3q + 8(1 - q) = 7q + 0(1 - q) \quad \Leftrightarrow \quad q = \frac{2}{3}$$

Theorem — Every **finite** strategic form game has **at least** one Nash equilibrium

Theorem — Generically, finite strategic form games have an odd number of Nash equilibria

example – 2×2 game