Repeated Games

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Reading assignments: Watson, Ch. 22 & 23

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"Love and duty are not the cement of modern societies...

The mechanism is reciprocity. Seemingly altruistic behavior, based on versions of the I'll-scratch-your-back-if-you-scratch-mine principle, require no nobility of spirit. Greed and fear will suffice as motivations: greed for the fruits of cooperation, and fear for the consequences of not reciprocating the cooperative overtures of others...

[A selfish rational agent] may not be an attractive individual, but he can cooperate very effectively with others like himself."

— Ken Binmore (1994) Game Theory and the Social Contract Vol. I

"Quis custodiet ipsos custodes?"

— Juvenal, Satire VI

- It is common that agents interact repeatedly
- Strategies can condition present choices on past behavior
- Anna could play according to the strategy "*I'll be nice to you as long as you are nice to me*"
- Each agent must consider two things
  - The direct consequences of his actions (payoffs)
  - How his/her present choices might influence those of other players
- Other players might be nice to Anna in the present because they want her to be nice to them in the future
- It is sometimes possible to generate incentives to implement desirable outcomes

repeated games

- Agents play a strategic form game various times in succession over a number of periods
- Their total payoff is the (possibly discounted) sum of the payoffs they get each period

- The strategic game being played each round is called the stage game
- The whole game consisting of the repetition of the stage game is called the supergame

- We will analyze the set of SPNE of the supergame

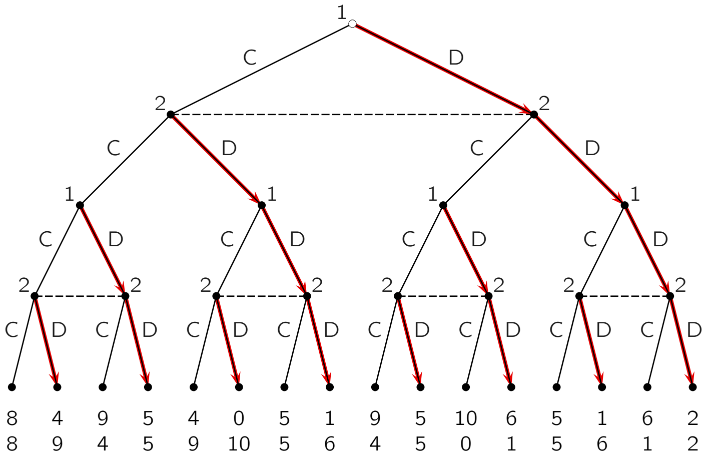
finitely repeated prisoners' dilemma

- Anna and Bob play the following prisoner's dilemma twice

	C	D
C	4 , 4	0 , 5
D	5 , 0	1 , 1

- They play the game once
- Then they play it again *after observing the outcome of the first period*
- The total payoff of each player is the sum of his/her payoffs across periods

finitely repeated prisoner's dilemma



finitely repeated prisoners' dilemma

- Subgame perfection $\Rightarrow$ both players defect on the last round
  - Because (D, D) is the only SPNE of the corresponding subgame

- It is optimal for each player to defect on the one-to-last round
  - Because the choices of the last round are independent of the past

- Players will defect every period in the unique SPNE
- Similar argument yields the same conclusion regardless of the number of rounds

> **Proposition** — If the stage game has a unique NE, and the number of rounds is finite, then the supergame has a unique SPNE which consists of playing the NE of the stage game in all subgames

example – a 3×3 game played twice

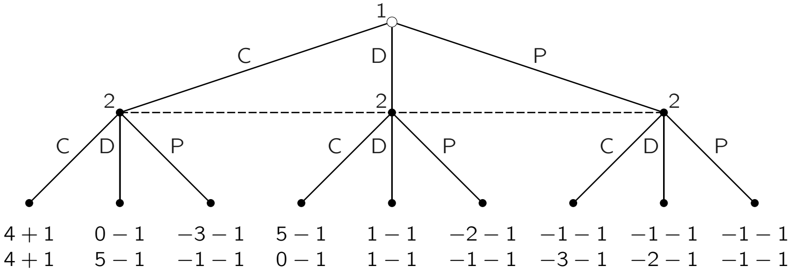
- Anna and Bob play the following stage game twice

	C	D	P
C	4 , 4	0 , 5	−3 , −1
D	5 , 0	1 , 1	−2 , −1
P	−1 , −3	−1 , −2	−1 , −1

- The corresponding game tree is already too big to be useful
- (How many terminal nodes are there?)

is cooperation possible in the example?

- The players could agree to play different continuation strategies on the second period, depending on the outcome of the first period
- For instance they could choose to play D on the second period if and only if the outcome of the first period was (C,C), and play P otherwise
- Note that these strategies induce NE on every second-period subgame
- Which NE is played depends on the first-period outcome



- The total value (v) of choosing an action on the first stage consists on payoff from the current period (u) plus continuation value (w) that will result from all future periods
- The continuation values induced by the continuation strategies from the previous slide are

	C	D	P
C	$1, 1$	$-1, -1$	$-1, -1$
D	$-1, -1$	$-1, -1$	$-1, -1$
P	$-1, -1$	$-1, -1$	$-1, -1$

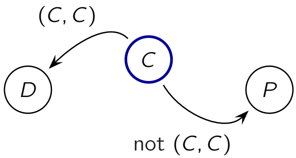
- The total value of each first-period outcome is obtain by adding the previous table to the payoff table of the stage game (why?)

	C	D	P
C	5 , 5	−1 , 4	−4 , −2
D	4 , −1	0 , 0	−3 , −2
P	−2 , −4	−2 , −3	−2 , −2

- Note that (C, C) is a NE of the game with payoffs $v = u + w$
- Choosing the right continuation strategies can make w depend on the current choices in the right way as to generate incentives for desirable behavior

- What we have shown is that the following strategies are a SPNE of the supergame (why?)

  - Play C on the 1st period
  - Play D on the 2nd period if the outcome of the first period was (C,C)
  - Play P in the second period otherwise



- This SPNE implements (C, C) on the first period even though C is dominated in the stage game!

- Infinite sequence of period indexed by $t = 0, 1, 2, 3, \ldots$
- On each period, player play a simultaneous move game, the stage game
- Past outcomes are publicly observable (perfect monitoring)
- Players discount future payoffs with a common discount factor $\delta \in (0, 1)$

$$v_i\big(\{s_t\}\big) = \sum_{t=0}^{\infty} \delta^t u_i(s_t) = u_i(s_0) + \delta u_i(s_1) + \delta^2 u_i(s_2) + \delta^3 u_i(s_3) + \ldots$$

- Interpretations of the discount factor
  - Firms paying interest $r \geq 0$ with $\delta = 1/(1 + r)$
  - Uncertainty about the end of the game with hazard rate δ
  - Overlapping generations with concern about the future

- The present value of a constant stream of payoffs $u_t = \bar{u}$ equals

$$v = \left(\frac{1}{1-\delta}\right)\bar{u}$$

- To see this, note that

$$\left.\begin{aligned} v &= \bar{u} + \delta\bar{u} + \delta^2\bar{u} + \delta^3\bar{u} + \dots \\ \delta v &= \delta\bar{u} + \delta^2\bar{u} + \delta^3\bar{u} + \delta^4\bar{u} + \dots \end{aligned}\right\} \quad \Rightarrow \quad (1-\delta)v = \bar{u}$$

- Suppose that an investment generates the stream of payoffs

$$-50, 2, 20, 5, 5, 5, 5, \dots$$

- The present value of the investment given $\delta = 0.9$ is

$$\begin{aligned} v &= -50 + \delta 2 + \delta^2 20 + \delta^3 5 + \delta^4 5 + \delta^5 5 + \dots \\ &= -50 + \delta 2 + \delta^2 20 + \delta^3 \left(5 + \delta 5 + \delta^2 5 + \dots\right) \\ &= -50 + \delta 2 + \delta^2 20 + \delta^3 \sum_{t=0}^{\infty} \delta^t 5 \\ &= -50 + \delta 2 + \delta^2 20 + \delta^3 \left(\frac{1}{1-\delta}\right) 5 \\ &= -50 + \frac{9}{10} 2 + \frac{81}{100} 20 + \frac{729}{1000} \frac{10}{1} 5 = 4.45 \end{aligned}$$

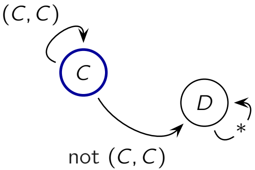
infinitely repeated prisoners' dilemma

- Suppose that Anna and Bob play our prisoner's dilemma repeatedly with discount factor $\delta = 0.5$

- The present value for each player if both play C forever of if both players play D forever are

$$\frac{1}{1-\delta}4 = 8 \qquad \frac{1}{1-\delta}1 = 2$$

- Consider the following "grim trigger" strategy
  - As long as everybody has played C in the past, play C
  - If at least one person has played D in the past, play D



- Suppose both players use grim trigger strategies and no one has defected
- If both players cooperate in the current period, they will both play *C* forever generating continuation values of 8
- If at least one player deviates by defecting in the current period, they will both play *D* forever generating continuation values of 2
- The total value for the current-period's actions is thus

value

	C	D
C	12 , 12	2 , 8
D	8 , 2	3 , 3

=

stage payoffs

	C	D
C	4 , 4	0 , 5
D	5 , 0	1 , 1

+

continuation values

	C	D
C	8 , 8	2 , 2
D	2 , 2	2 , 2

- Suppose both players use grim trigger strategies and someone has defected at least once in the past
- Then players will defect forever after generating continuation values of 2
- The choices in the current period will not change that, but they can still change the payoffs in the current period if a player decides to cooperate instead
- The total value for the current-period's actions is thus

value

	C	D
C	6 , 6	0 , 7
D	7 , 0	3 , 3

=

stage payoffs

	C	D
C	4 , 4	0 , 5
D	5 , 0	1 , 1

+

continuation values

	C	D
C	2 , 2	2 , 2
D	2 , 2	2 , 2

grim trigger as a spne

- We have shown two things
  - In histories after which players using grim trigger strategies are supposed to cooperate, (C, C) is part of a NE of the corresponding subgame
  - In histories after which players using grim trigger strategies are supposed to defect, (D, D) is part of a NE of the corresponding subgame
- This implies that the grim trigger strategies constitute a SPNE of our infinitely repeated prisoners' dilemma with $\delta = 0.5$