Risk*

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Winter 2020

In many economic applications, the relevant uncertain outcomes are monetary prizes. In such settings, it is natural to assume that agents prefer higher returns and lower risk. The first part of these notes defines what it means to dislike risk, and proposes a way to measure the risk attitudes of economic agents. It also briefly discusses the behavior of risk averse agents facing insurance and investment problems. The second part of the notes proposes ways of measuring the size and riskiness of lotteries. Throughout the document, we maintain the expected utility hypothesis.

1. Risk Aversion

This section studies a way to define and measure risk aversion due to Pratt (1964) and Arrow (1965). Consider an economic agent who makes choices to maximize the expectation of a Bernoulli utility function u. I assume u is bounded above. The set of outcomes is $X = \mathbb{R}$. Uncertain prospects can thus be modeled as random variables $\mathbf{x}$. I use the notation $\mathbf{x} \sim F$ to denote that F is the cumulative distribution function (c.d.f.) of $\mathbf{x}$, i.e., $F(x) = \Pr(\mathbf{x} \leq x)$ for $x \in \mathbb{R}$.

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*Lecture notes for Econ 9602 at Western University.

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1.1. Risk-Averse Individuals

An economic agent is called *risk averse* if they always prefer the expected value of a lottery to the lottery itself, i.e., if

$$u(\mathbb{E}[\,\mathbf{x}\,]) \geq \mathbb{E}[\,u(\mathbf{x})\,], \tag{1}$$

for every lottery $\mathbf{x}$ with finite expectation. Equation (1) is called Jensen's inequality. Risk aversion is actually equivalent to money having decreasing marginal utility, as in the Cramer-Bernoulli resolution to the St. Petersburg Paradox. In mathematical terms, decreasing marginal utility means concavity.

Proposition 1.1 *An agent is risk averse if and only if his Bernoulli utility function is concave.*

Proof. ($\Rightarrow$) Suppose that (1) holds for all random variables with finite expectation, and fix any $x, y \in \mathbb{R}$ and any $\mu \in [0,1]$. Consider a random variable $\mathbf{x}$ that takes the value x with probability μ and the value y with probability $(1-\mu)$. Condition (1) implies that

$$u\Big(\mu x + (1-\mu)y\Big) = u(\mathbb{E}[\,\mathbf{x}\,]) \geq \mathbb{E}[\,u(\mathbf{x})\,] = \mu u(x) + (1-\mu)u(y).$$

Hence, u is concave.

($\Leftarrow$) Now, suppose that u is concave. We will prove that (1) holds for every random variable with finite support. The proof is by mathematical induction on the cardinality of the support. For (degenerate) random variables with only one point on their support, (1) holds trivially. Now, suppose that (1) holds for all random variables with at most n points on their support, and let $\mathbf{x}$ be a random variable with $n+1$ point on their support. In particular, suppose that $\mathbf{x}$ takes the values $x_1, \ldots, x_{n+1}$ with strictly positive probabilities $p_1, \ldots, p_{n+1}$, respectively.

Let $q = (q_1, \ldots, q_n)$ be given by $q_i = p_i/(1-p_{n+1})$. It is straightforward to show that q is a well-defined probability vector using the facts that $\sum_{i=1}^{n} p_i = 1$ and $p_i \in (0,1)$ for $i = 1, \ldots, n+1$. Note that

$$\sum_{i=1}^{n+1} p_i u(x_i) = p_{n+1}u(x_{n+1}) + \sum_{i=1}^{n}(1-p_{n+1})\frac{p_i}{1-p_{n+1}}u(x_i)$$

$$
\begin{aligned}
&= p_{n+1}u(x_{n+1}) + (1-p_{n+1})\sum_{i=1}^{n} q_i u(x_i) \\
&\leq p_{n+1}u(x_{n+1}) + (1-p_{n+1})u\left(\sum_{i=1}^{n} q_i x_i\right) \\
&\leq u\left(p_{n+1}x_{n+1}) + (1-p_{n+1})\sum_{i=1}^{n} q_i x_i\right) = u\left(\sum_{i=1}^{n+1} p_i x_i\right),
\end{aligned}
$$

where the first inequality follows from the induction hypothesis, and the second one from the concavity of u. ■

Jensen's inequality holds for general random variables, as long as their expected value is finite. One way to prove the general case is using a separating hyperplane argument. See for instance this Stack Exchange answer.

Another way to think of risk aversion is in terms of the maximum prize a decision maker would pay for a lottery. Fix a lottery $\mathbf{x}$ and a continuous and strictly increasing Bernoulli utility function u. The *certainty equivalent* of $\mathbf{x}$ given u is the number number $c_u(\mathbf{x})$ defined as the unique solution to

$$
u\Big(c_u(\mathbf{x})\Big) = \mathbb{E}[\, u(\mathbf{x})\,]. \tag{2}
$$

Note that $\inf_{x \in \operatorname{supp} \mathbf{x}} u(x) \leq \mathbb{E}[\, u(\mathbf{x})\,] \leq \sup_{x \in \operatorname{supp} \mathbf{x}} u(\mathbf{x})$, where $\operatorname{supp} \mathbf{x}$ denotes the support of $\mathbf{x}$. Hence, the mean-value theorem implies that (2) has a solution. The solution is unique because u is strictly increasing.

Proposition 1.2 *An agent with an increasing and bounded Bernoulli utility function u is risk averse if and only if $c_u(\mathbf{x}) \leq \mathbb{E}[\,\mathbf{x}\,]$ for every random variable $\mathbf{x}$.*

Proof. Since u is increasing and $u(c_u(\mathbf{x})) = \mathbb{E}[\, u(\mathbf{x})\,]$, we have

$$
\mathbb{E}[\, u(\mathbf{x})\,] \leq u(\mathbb{E}[\,\mathbf{x}\,]) \iff c_u(\mathbf{x}) \leq \mathbb{E}[\,\mathbf{x}\,]
$$

for every random variable $\mathbf{x}$. The left-hand inequality corresponds to the definition of risk aversion, and the right-hand inequality is the condition from the proposition. ■

1.2. Measuring Risk Aversion

Consider an economic agent with a twice continuously differentiable, strictly increasing, and concave utility function u. Since risk-aversion is related to curvature, a natural candidate to measure risk aversion is the second derivative u''. However, a problem arises because Bernoulli utility functions are only unique up to affine transformations. For any positive constant $a > 0$, $a \cdot u$ represents exactly the same preferences as u. However, if $a \neq 1$, then $(a \cdot u)'' = a \cdot u'' \neq u''$.

One way to deal with this issue is to use the index proposed by Arrow (1965) and Pratt (1964). The *Arrow-Pratt index of absolute risk aversion* of u at wealth level x is the number $r_u(x)$ given by

$$r_u(x) = -\frac{u''(x)}{u'(x)}.$$

Note that r_u is invariant with respect to affine transformations. Hence, it depends only on preferences. Also, it is consistent with Jensen's inequality in that risk averse agents have positive risk aversion. Finally, it can be justified by the following theorem

Theorem 1.3 (Pratt (1964)) *Given two strictly increasing, bounded above, and twice continuously differentiable Bernoulli utility functions u and v, the following are equivalent:*

i. $r_u(x) \geq r_v(x)$ *for all* $x \in \mathbb{R}$.

ii. There exists a monotone concave function g such that $u = g \circ v$.

iii. $c_u(\mathbf{x}) \leq c_v(\mathbf{x})$ *for every lottery* $\mathbf{x}$ *with finite expected value.*

Proof. ($i \Leftrightarrow ii$) Since v is strictly increasing, it is invertible and v^{-1} is also strictly increasing. Let $g = u \circ v^{-1}$. Since u and v^{-1} are strictly increasing and differentiable, so is g. It remains to show that g is concave if and only if $r_u(x) \geq r_v(x)$ for all $x \in \mathbb{R}$.

Taking derivatives of $u = g \circ v$ with the chain rule yields $u' = (g' \circ v) \cdot v'$ and $u'' = (g'' \circ v) \cdot (v')^2 + (g' \circ v) \cdot v''$. Pointwise dividing u'' by $u' > 0$ yields

$$\frac{u''}{u'} = \frac{(g'' \circ v) \cdot (v')^2}{(g' \circ v) \cdot v'} + \frac{(g' \circ v) \cdot v''}{(g' \circ v) \cdot v'} \implies r_u - r_v = \left(\frac{(v')^2}{(g' \circ v) \cdot v'} \right) \cdot (g'' \circ v)$$

Hence, for each $x \in \mathbb{R}$, $r_u(x) \geq r_v(x)$ if and only if $g''(v(x)) \leq 0$.

The equivalence between (*iii*) and (*ii*) is left as an exercise for the reader. ■

An important class of utility function are those that exhibit *constant absolute risk aversion* (CARA). These utility functions are important because they form a single-parameter class, and the parameter has a clear structural interpretation ideal for comparative statics. Hence, they are ubiquitous in applied work. Setting r_u equal to a constant α results in the differential equation

$$u''(x) = -\alpha u'(x).$$

Using the change of variables $v = u'$ we can rewrite the equation as

$$\frac{dv}{dx} = -\alpha v \implies \int \frac{1}{v}\, dv = -\alpha \int dx \implies \log(v) = -\alpha x + c_1,$$

where c is an integration constant. Applying the exponential function to both sides and undoing the change of variables yields

$$u'(x) = c_2 \exp(-\alpha x) \implies u(x) = c_3 \exp(-\alpha x) + c_4.$$

Since Bernoulli utility functions are unique up to affine transformations, we can choose c_3 and c_4 freely. The only restriction is that we must have $c_3 < 0$ so that u is increasing in wealth. The standard approach is to set $c_4 = 0$ and $c_3 = -1$ to get $u(x) = -\exp(-\alpha x)$.

There are other important measures of risk aversion. For example, a common assumption is that wealthier individuals are more tolerant to risk. Hence, in some applications it is useful to have a measure of risk attitudes that controls for the level of wealth. The *Arrow-Pratt coefficient of relative* risk aversion is one such measure. It is given by $\rho_u(x) = x \cdot r_u(x)$. In applications involving dynamics, the saving behavior of households actually depends on the sign third derivative of the utility function. Another example is the notion of *prudence* defined by $-u'''(x)/u''(x)$, which helps to characterize precautionary saving motives (Kimball, 1990).

1.3. Applications

Risk aversion plays an important role in many applications. Two of them are investment, because financial assets often have risky returns. Another one is insurance, because risk averse agents will try to buy instruments that reduce their exposure to risk. In this section, we will consider two simple examples to illustrate these points. Consider the problem of an investor with an initial wealth $\omega > 0$ and a bounded-above, concave, strictly increasing, and twice continuously differentiable Bernoulli utility function u.

First, suppose that the individual can invest in a risky asset with a random return $\mathbf{x}$ such that $\mathbb{E}[\,\mathbf{x}\,] > 0$. That is, if the investor invests a units in the asset, the asset will pay $a(1+\mathbf{x})$. The investor is subject to portfolios satisfying $0 < a < \omega$.

Proposition 1.4 *A risk-averse investor would invest a positive amount in an asset as long as it has a positive expected return.*

Proof. The expected utility from investing a is given by $U(a) := \mathbb{E}[\,u(\omega + a\mathbf{x})\,]$. The optimal investment maximizes $U(a)$ subject to the constraints $0 \leq a \leq \omega$. The necessary first-order condition for 0 to be optimal is $U'(0) \leq 0$. Note that $U'(a) = \mathbb{E}[\,u'(\omega + a\mathbf{x})\mathbf{x}\,]$. Hence, $U'(0) = \mathbb{E}[\,u'(\omega)\mathbf{x}\,] = u'(\omega)\mathbb{E}[\,\mathbf{x}\,] > 0$. ■

Now suppose that the individual faces the possibility of a random loss of magnitude $\mathbf{y}$. Suppose that $\mathbf{y} < 0$ almost surely. An insurance company offers an insurance policy that mitigates the losses. More precisely, the individual can choose a number $a \in [0,1]$ and, in case of a loss, the insurance company will pay $a\mathbf{y}$. If the unit price of the policy is $p > 0$, then the final wealth of the individual woudl be given by $w - pa - (1-a)\mathbf{y}$. We say that the insurance is actuarially fair if there are ero profits, that is $p = \mathbb{E}[\,\mathbf{y}\,]$.

Proposition 1.5 *A risk averse individual offered actuarially fair insurance will insure completely.*

Proof. For $a \in [0,1]$, let $\mathbf{z}(a)$ denote the final wealth of an individual who chooses to insure a fraction a of their losses. Note that having actuarially fair insurance implies that

$$\mathbb{E}[\,\mathbf{z}(a)\,] = \mathbb{E}[\,w - \mathbf{y} + a(\mathbf{y} - p)\,] = w - \mathbb{E}[\,\mathbf{y}\,]\,.$$

Now fix some $a \in [0,1]$. The wealth of an individual who insures completely equals $\mathbf{z}(1) = w - p = w - \mathbb{E}[\,\mathbf{y}\,] = \mathbb{E}[\,\mathbf{z}(a)\,]$. Since the individual is risk averse, $\mathbb{E}[\,u(\mathbf{z}(a))\,] \leq u(\mathbb{E}[\,\mathbf{z}(a)\,]) = \mathbb{E}[\,u(\mathbf{z}(1))\,]$. ■

2. Comparing Distributions

This section presents different ways of comparing lotteries over monetary outcomes. First, we introduce two ordinal rankings called first-order and second-order stochastic dominance. There orders informally capture the idea that one lottery offers larger prizes than another, or that one lottery is less risky than another one, respectively. The stochastic dominance relations are incomplete. The last part of the section introduces a complete ranking due to Aumann and Serrano (2008). For simplicity, some of the discussion restricts attention to lotteries with compact support. However, the results hold true in more general spaces.

The terms first and second-order stochastic dominance were introduced by Hadar and Russell (1969) and Hanoch and Levy (1969). The notions were developed independently by Rothschild and Stiglitz (1970), who also came up with a useful characterization in terms of mean-preserving spreads. There is a notion of third-degree stochastic dominance introduced by Whitmore (1970). The main proofs and techniques are related to an old statistics literature from the 1930s regarding a topic called majorization. For a review of this work see Levy (1992).

2.1. First-Order Stochastic Dominance

Suppose that lottery $\mathbf{x}$ pays 1 with probability 0.5 and 0 with probability 0.5, and lottery $\mathbf{y}$ is uniformly distributed on $[-1,1]$. Intuitively, lottery $\mathbf{x}$ appears to be "better" than $\mathbf{y}$. This section introduces a ranking over lotteries that formalizes that comparison. The ranking reflects the preferences of *all* expected utility maximizers with monotone Bernoulli utility functions.

Definition 2.1 Given two random variables $\mathbf{x}$ and $\mathbf{y}$, we say that $\mathbf{x}$ *first-order stochastically dominates* $\mathbf{y}$, and denote it by $\mathbf{x} \geq_1 \mathbf{y}$, if and only if $\mathbb{E}[\,u(\mathbf{x})\,] \geq \mathbb{E}[\,u(\mathbf{y})\,]$ for every non-decreasing function $u : \mathbb{R} \to \mathbb{R}$.

It is straightforward to verify that first-order stochastic dominance is reflective, transitive, but not complete. The following proposition offers a straightforward way to determine whether $\mathbf{x}$ first-order stochastically dominates $\mathbf{y}$ by comparing their c.d.f.s.

Proposition 2.1 *Given two random variables $\mathbf{x} \sim F$ and $\mathbf{y} \sim G$ with supports contained in $[0, 1]$, $\mathbf{x} \geq_1 \mathbf{y}$ if and only if $F(\xi) \leq G(\xi)$ for every $\xi \in \mathbb{R}$.*

Before proving the proposition, let us try to build some intuition. Recall that $F(\xi)$ and $G(\xi)$ measure the probabilities of $\mathbf{x} \leq \xi$ and $\mathbf{y} \leq \xi$, respectively. Hence, the condition $F(\xi) \leq G(\xi)$ says that it is more likely for $\mathbf{y}$ to give a prize smaller than ξ. If this is the case for every ξ, then it is natural for agents who like money to prefer $\mathbf{x}$ over $\mathbf{y}$. Going back to our motivating example, the c.d.f.s of $\mathbf{x}$ and $\mathbf{y}$ are given by

$$\Pr(\mathbf{x} \leq \xi) = \begin{cases} 0 & \text{if} \quad \xi < 0 \\ \dfrac{1}{2} & \text{if} \quad 0 \leq \xi < 1 \\ 1 & \text{if} \quad \xi \geq 1 \end{cases}, \qquad \Pr(\mathbf{y} \leq \xi) = \begin{cases} 0 & \text{if} \quad \xi < -1 \\ \dfrac{1-\xi}{2} & \text{if} \quad 0 \leq \xi < 1 \\ 1 & \text{if} \quad \xi \geq 1 \end{cases}.$$

Hence, $\Pr(\mathbf{y} \leq \xi) \geq \Pr(\mathbf{x} \leq \xi)$ for all $\xi \in \mathbb{R}$.

Proof of Proposition 2.1. ($\Rightarrow$) Suppose that $\mathbf{x} \geq_1 \mathbf{y}$. For each $\xi \in [0, 1]$, let $u_\xi(x) = 1 - \mathbb{1}(x \leq \xi)$, where $\mathbb{1}(\,\cdot\,)$ denotes the indicator function. The result follows from the fact that $\mathbb{E}[\,u_\xi(\mathbf{x})\,] = F(\xi)$ and $\mathbb{E}[\,u_\xi(\mathbf{y})\,] = G(\xi)$ for all $\xi \in [0, 1]$.

($\Leftarrow$) I will only consider continuously differentiable utility functions. For a general proof see this Stack Exchange answer. Suppose that $F(\xi) \leq G(\xi)$ for every $\xi \in \mathbb{R}$, and fix any continuously differentiable and non-decreasing function $u : \mathbb{R} \to \mathbb{R}$. Integrating by parts yields

$$\int_0^1 u(\xi)\, d(F(\xi) - G(\xi)) + \int_0^1 u'(\xi)(F(\xi) - G(\xi))\, d\xi = u(\xi)(F(\xi) - G(\xi))\Big|_0^1.$$

The right hand side is equal to 0 because $F(1) = G(1)$ and $F(0) = G(0)$. Hence, we have that

$$\mathbb{E}[\,u(\mathbf{x})\,] - \mathbb{E}[\,u(\mathbf{y})\,] = -\int_0^1 u'(\xi)(F(\xi) - G(\xi))\,d\xi.$$

Since $u'(\xi) \geq 0$ and $F(\xi) \leq G(\xi)$ for every $\xi \in \mathbb{R}$, the right-hand side is non-negative, and therefore $\mathbb{E}[\,u(\mathbf{x})\,] \geq \mathbb{E}[\,u(\mathbf{y})\,]$. ■

2.2. Second-Order Stochastic Dominance

Consider two lotteries $\mathbf{x}$ and $\mathbf{y}$. Suppose that $\mathbf{y}$ depends on both the realization of $\mathbf{x}$ and the flip of a fair coin independent of $\mathbf{x}$. If the coin lands head, $\mathbf{y}$ pays $\mathbf{x}+1$, otherwise it pays $\mathbf{x}-1$. Both $\mathbf{x}$ and $\mathbf{y}$ have the same expected value, but $\mathbf{y}$ has an added layer of "risk". Hence, risk-averse individuals should prefer $\mathbf{x}$ to $\mathbf{y}$. This idea is captured by the notion of second-order stochastic dominance.

Definition 2.2 Given two random variables $\mathbf{x}$ and $\mathbf{y}$, we say that $\mathbf{x}$ *second-order stochastically dominates* $\mathbf{y}$, and denote it by $\mathbf{x} \geq_2 \mathbf{y}$, if and only if $\mathbb{E}[\,u(\mathbf{x})\,] \geq \mathbb{E}[\,u(\mathbf{y})\,]$ for every non-decreasing and *concave* function $u : \mathbb{R} \to \mathbb{R}$.

Second-order stochastic dominance is consistent with first-order stochastic dominance in that $\mathbf{x} \geq_1 \mathbf{y}$ implies $\mathbf{x} \geq_2 \mathbf{y}$. Moreover, $\geq_2$ is reflexive and transitive, but incomplete. We will analyze two different characterizations of second-order stochastic dominance. The first characterization is in terms of mean-preserving spreads, defined as follows.

Definition 2.3 Given two random variables $\mathbf{x}$ and $\mathbf{y}$, we say that $\mathbf{y}$ is a *mean-preserving spread* of $\mathbf{x}$ if and only if $\mathbb{E}[\,\mathbf{y} - \mathbf{x}|\mathbf{x}\,] = 0$ almost surely.

Note that, if $\mathbf{y}$ is a mean-preserving spread of $\mathbf{x}$, then $\mathbb{E}[\,\mathbf{y}\,] = \mathbb{E}[\,\mathbf{x}\,]$. Intuitively, $\mathbf{y}$ is a mean preserving spread of $\mathbf{x}$ if it can be constructed by adding noise or risk to $\mathbf{x}$. That is, if we can write it as $\mathbf{y} = \mathbf{x} + \mathbf{z}$, where $\mathbf{z}$ is a random variable such that $\mathbb{E}[\,\mathbf{z}|\mathbf{x}\,] = 0$. In our motivating example, $\mathbf{z}$ would be a random variable that takes the value 1 if the coin lands heads, and -1 if it lands tails.

The second characterization is in terms of c.d.f.s. Intuitively, if $\mathbf{x} \sim F$ and $\mathbf{y} \sim G$ have the same expectation but $\mathbf{x} \geq_2 \mathbf{y}$, then it must be the case that

G assigns more probability to the tails of the distribution than F. The two characterizations are formalized in the following proposition.

Proposition 2.2 *Given two random variables* $\mathbf{x} \sim F$ *and* $\mathbf{y} \sim G$ *with supports contained in* $[0,1]$ *and such that* $\mathbb{E}[\,\mathbf{x}\,] = \mathbb{E}[\,\mathbf{y}\,]$, *the following are equivalent*

(i) $\mathbf{x}$ *second-order stochastically dominates* $\mathbf{y}$.

(ii) There exist random variables $\mathbf{x}' \sim F$ *and* $\mathbf{y}' \sim G$ *such that* $\mathbf{y}$ *is a mean-preserving spread of* $\mathbf{x}$.

(iii) For every $\xi \in [0,1]$

$$\int_0^\xi F(\xi)\, d\xi \leq \int_0^\xi G(y)\, dy.$$

I will only prove that (*ii*) implies (*i*). The equivalence between (*i*) and (*iii*) can be established using integration by parts twice. It is left as an exercise for the reader. Showing that either (*i*) or (*iii*) imply (*ii*) is somewhat more involved, and the proof is hard to find online.

Proof that (ii) implies (i). Suppose that $\mathbf{y}$ is a mean preserving spread of $\mathbf{x}$, and u is concave. Let $\mathbf{z} = \mathbf{y} - \mathbf{x}$. Then,

$$\mathbb{E}[\,u(\mathbf{y})\,] = \mathbb{E}[\,\mathbb{E}[\,u(\mathbf{x}+\mathbf{z})|\mathbf{x}\,]\,] \leq \mathbb{E}[\,u\,(\mathbb{E}[\,\mathbf{x}+\mathbf{z}|\mathbf{x}\,])\,] = \mathbb{E}[\,u\,(\mathbf{x})\,]\,,$$

where the first equality follows from the law of iterated expectations, the inequality from Jensen's inequality, and the second equality from te facts that $\mathbb{E}[\,\mathbf{x}|\mathbf{x}\,] = \mathbf{x}$ and $\mathbb{E}[\,\mathbf{z}|\mathbf{x}\,] = 0$. ∎

2.3. An Index of Riskiness

First-order stochastic dominance captures the preferences of all expected utility maximizers who prefer more money. Second-order stochastic dominance captures the preferences of those who, in addition, dislike risk. Neither ranking is complete. One way to obtain a complete ranking of distributions is to restrict attention to a smaller class of decision makers and a smaller class of lotteries. Aumann and Serrano (2008) propose a ranking that summarizes the preferences of all CARA agents over a class of lotteries called gambles.

A *gamble* is a random variable $\mathbf{x}$ such that $\mathbb{E}[\,\mathbf{x}\,] > 0$ and $\Pr(\mathbf{x} < 0) > 1$. Gambles are good investments on average, but are bad investments with positive probability. Fix a given gamble $\mathbf{x}$. Since gambles have positive expected value, risk-neutral individuals would always accept the gamble. We can thus expect that an agent who is only slightly risk averse would also accept it. On the other hand, a decision maker who is very risk averse would find the prospect of losing money and would reject the gamble. Aumann and Serrano (2008) propose to rank gambles based on the level of risk aversion that would make a CARA agent indifferent between accepting the gamble or not.

The *Aumann-Serrano index of risk aversion* of gamble $\mathbf{x}$ is denoted by and is defined to be the unique solution $R_{\mathbf{x}}$ to the equation

$$\mathbb{E}\left[-\exp\left(-\frac{\mathbf{x}}{R}\right)\right] = -1.$$

The left-hand side of the equation is the expected utility from accepting $\mathbf{x}$ for a CARA agent with risk aversion equal to R. The right-hand side is the expected utility of rejecting it, i.e., $u(0) = -1$. It is easy to show that the left-hand side ranges from 0 to infinity as a function of R. Moreover, Lemma 2.3 below implies that it is strictly increasing in R. Hence, the equation has a unique solution and the Aumann-Serrano index is well defined.

Lemma 2.3 *Let u_α and u_β be CARA utility functions with coefficients of risk aversion α and β, respectively. If $\alpha > \beta$, then $\mathbb{E}[\,u_\alpha(\mathbf{x})\,] < \mathbb{E}[\,u_\beta(\mathbf{x})\,]$ for every random variable $\mathbf{x}$.*

Proof. Suppose that $\alpha > \beta > 0$. Let v_α and v_β be positive affine transformations of u_α and u_β such that $v_\alpha(0) = v_\beta(0) = 1$ and $v'_\alpha(0) = v'_\beta(0) = 1$. Recall that the Arrow-Pratt index of risk aversion is invariant with respect to positive affine transformations. This implies that $r_{v_\alpha}(\xi) = \alpha$ and $r_{v_\beta}(\xi) = \beta$ for all $\xi \in \mathbb{R}$.

For every $x > 0$ we have that

$$\begin{aligned}\log(v'_\alpha(x)) &= \log(v'_\alpha(x)) - \log(v'_\alpha(0)) = \int_0^x \Big[log(v'_\alpha(\xi))\Big]'\,d\xi \\ &= \int_0^x \frac{v''_\alpha(\xi)}{v'_\alpha(\xi)}\,d\xi = -\int_0^x \alpha\,d\xi < -\int_0^x \beta\,d\xi = \log(v'_\beta(x)).\end{aligned}$$

Since $\log(\,\cdot\,)$ is strictly increasing, it follows that $v'_\alpha(x) < v'_\beta(x)$ for $x > 0$.

Therefore, for every $x > 0$ we have

$$v_\alpha(x) = v_\alpha(x) - v_\alpha(0) = \int_0^x v'_\alpha(\xi)\, d\xi < \int_0^x v'_\beta(\xi)\, d\xi = v_\beta(x).$$

An similar argument can be used to show that $v_\alpha(x) < v_\beta(x)$ also for $x < 0$. Hence, $\mathbb{E}[\, v_\alpha(\mathbf{x})\,] < \mathbb{E}[\, v_\beta(\mathbf{x})\,]$ for every random variable $\mathbf{x}$. The conclusion of the proposition then follows because v_α and v_β are positive affine transformations of u_α and u_β. ■

The Aumann-Serrano index provides a complete and transitive ranking of gambles. Moreover, the Aumann-Serrano ranking is consistent with first-order and second-order stochastic dominance.

Proposition 2.4 *Given gambles* $\mathbf{x}$ *and* $\mathbf{y}$*, if* $\mathbf{x} \geq_1 \mathbf{y}$ *or* $\mathbf{x} \geq_2 \mathbf{y}$*, then* $R_\mathbf{x} \leq R_\mathbf{y}$*.*

Proof. Suppose that $\mathbf{x} \geq_1 \mathbf{y}$ or $\mathbf{x} \geq_2 \mathbf{y}$, and let u be a CARA utility function with coefficient of absolute risk aversion equal to $R_\mathbf{y}$. Since u is nondecreasing and concave, it follows that $\mathbb{E}[\, u(\mathbf{x})\,] \geq \mathbb{E}[\, u(\mathbf{y})\,] = u(0)$. That is,

$$\mathbb{E}\left[-\exp\left(-\frac{\mathbf{x}}{R_\mathbf{y}} \right) \right] \geq -1.$$

Lemma 2.3 implies that the left-hand side is decreasing in the degree of risk aversion. Therefore, $R_\mathbf{x} \leq R_\mathbf{y}$. ■

2.4. Background Risk

We have been analyzing specific forms of risk in isolation. In reality, people face many sources of risk simultaneously. Pomatto et al. (2019) and Tarsney (2018) have shown that the presence of background risk can be very important.

Proposition 2.5 *Given random variables* $\mathbf{x}$ *and* $\mathbf{y}$ *with finite mean such that* $\mathbb{E}[\, x\,] \geq \mathbb{E}[\, y\,]$*, there exists a third random variable* $\mathbf{z}$ *independent of* $\mathbf{x}$ *and* $\mathbf{y}$ *and such that* $\mathbf{x} + \mathbf{z} \geq_1 \mathbf{y} + \mathbf{z}$*. Moreover, if* $\mathbf{z}' \geq_2 \mathbf{z}$ *and is also independent of* $\mathbf{x}$ *and* $\mathbf{y}$*, then* $\mathbf{x} + \mathbf{z}' \geq_1 \mathbf{y} + \mathbf{z}'$*.*

Think of $\mathbf{z}$ as the background risk that the agent faces. In the presence of sufficiently large background risk, any expected utility maximizer with monotone preferences would compare **x** and **y** just in terms of expectations, the way a risk-neutral decision maker would. This is an interesting and exciting result that opens up many questions, being that it is at odds with vast evidence that people are risk averse. The proof is constructive, but it is rather elaborate. An important thing to consider is that the random variable **z** has fat tails. However, it can be arbitrarily close to a Gaussian random variable.

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