Minkowski’s separating hyperplane theorem
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Theorem 1 For every pair of convex non-empty sets $X, Y \subseteq \mathbb{R}^n$ with disjoint interiors, there exists a non-null vector $z \in \mathbb{R}^n \setminus \{0\}$ and a scalar $\mu \in \mathbb{R}$ such that:

$$\sup\{z \cdot y | y \in Y\} \leq \mu \leq \inf\{z \cdot x | x \in X\}$$

The theorem follows from the two following lemmas.

**Figure (1)** Proof of Minkowski’s Separating Hyperplane Theorem

**Lemma 1** For every non-empty, closed and convex set $X \subseteq \mathbb{R}^n$ and every point $y \in \mathbb{R}^n \setminus X$, there exists a non-null vector $z \in \mathbb{R}^n \setminus \{0\}$ such that $z \cdot y > \sup\{z \cdot x | x \in X\}$.

**Proof.** Let $X \subseteq \mathbb{R}^n$ be non-empty and convex and fix a point $y \in \mathbb{R}^n \setminus X$. Since $X \neq \emptyset$ we can fix some arbitrary point $x_0 \in X$. Consider the compact non-empty set $W = \{w \in \mathbb{R}^n | \|y - w\|_2 \leq \|x_0 - y\|_2\}$ and let $X' = X \cap W$. In words, $X'$ is the set of points in $X$ that are as close to $y$ as $x_0$ according to the Euclidean metric. Clearly we have $x_0 \in X'$ and thus $X'$ is a compact non-empty set. Hence, by continuity of the Euclidean norm and Weierstrass’ theorem, we know that there exists some point $x^* \in \arg\min\{\|y - x\|_2 | x \in X'\}$. By construction, $x^*$ is the point of $X$ which is closest to $y$ according to the Euclidean metric. Let $z = y - x^* \in \mathbb{R}^n$, since $y \notin X$ we know that $z \neq 0$.

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Let $x \in X$ be an arbitrary point and let $v : (0, 1) \to \mathbb{R}^n$ be the function given by $v_\mu = \mu x + (1 - \mu)x^*$. By convexity of $X$ we know that $v \in X^{(0, 1)}$. Thus, by definition of $x^*$, $0 < \|y - x^*\|_2 \leq \|y - v_\mu\|_2$ for all $\mu \in [0, 1]$ and:

$$0 \geq \|y - x^*\|^2_2 - \|y - v_\mu\|^2_2$$

$$= z \cdot z - (y - \mu x + (1 - \mu)x^*) \cdot (y - \mu x + (1 - \mu)x^*)$$

$$= z \cdot z -(z + \mu (x^* - x)) \cdot (z + \mu (x^* - x))$$

$$= z \cdot z - z \cdot z - 2\mu z \cdot (x^* - x) - \mu^2 (x^* - x) \cdot (x^* - x)$$

$$= -2\mu z \cdot (x^* - x) - \mu^2 \|x^* - x\|_2$$

This implies that:

$$z \cdot (x^* - x) \geq -\frac{1}{2} \mu \|x^* - x\|_2 \xrightarrow{\mu \to 0} 0$$

Hence $z \cdot x^* \geq z \cdot x$. Since $x$ was arbitrary, this implies $z \cdot x^* = \max \{z \cdot x \mid x \in X\}$. Finally, notice that $0 < z \cdot z = z \cdot (y - x^*)$ and thus $z \cdot y > z \cdot x^*$. ■

**Lemma 2** For every non-empty and convex set $X \subseteq \mathbb{R}^n$ and every point $y^*$ in the boundary of $X$, there exists a non-null vector $z \in \mathbb{R}^n \setminus \{0\}$ such that $z \cdot y \geq \sup \{z \cdot x \mid x \in X\}$.

**Proof.** Let $X \subseteq \mathbb{R}^n$ be a non-empty convex set and consider a boundary point $y \in \text{fro}(X)$. By definition of boundary, there exists a sequence $(y_n)_{n \in \mathbb{N}} \in (\text{int}(\mathbb{R}^n \setminus \text{cl}(X)))^{\mathbb{N}}$ such that $\lim y_n = y$. By Lemma 1, there exists a sequence $(w_n)_{n \in \mathbb{N}} \in (\mathbb{R}^n \setminus \{0\})^{\mathbb{N}}$, such that $w_n \cdot y_n > \sup \{w_n \cdot x \mid x \in \text{cl}(X)\}$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ we have $w_n \neq 0$ and, consequently $\|w_n\|_2 > 0$. We can thus define the sequence $(z_n)_{n \in \mathbb{N}} \in B^{\mathbb{N}}$ given by $z_n = w_n/\|w_n\|_2$, where $B = \{v \in \mathbb{R}^n \mid \|v\|_2 = 1\}$ is the unit circle in $\mathbb{R}^n$. This transformation preserves the inequalities $z_n \cdot y_n > z_n \cdot x$ for all $x \in \text{cl}(X)$ and all $n \in \mathbb{N}$. Since $B$ is compact, we know that $(z_n)_{n \in \mathbb{N}}$ has a convergent subsequence converging to some limit $z \in B$. Since weak inequalities are preserve under limits of linear functions, we have $z \cdot y \geq z \cdot x$ for all $x \in \text{cl}(X)$. Consequently, since $X \subseteq \text{cl}(X)$, we have $z \cdot y \geq \sup \{z \cdot x \mid x \in X\}$. ■

Let $X, Y \subseteq \mathbb{R}^n$ be convex and have disjoint interiors and let $W = \text{int}(X) - \text{int}(Y) \subseteq \mathbb{R}^n$. Since $\text{int}(Y) \cap \text{int}(X) = \emptyset$, we know that $0 \notin W$. Simple algebra shows that $W$ is convex. From the previous lemmas it follows that there exists some $z \in \mathbb{R}^n \setminus \{0\}$ such that $0 = z \cdot 0 \geq z \cdot (x - y)$ for all $x \in X$ and all $y \in Y$ (if $0 \in \text{fro}(W)$ use Lemma 1, otherwise use Lemma 2). Which implies that $z \cdot x \geq z \cdot y$ for all $x \in X$ and all $y \in Y$, and hence $\sup \{z \cdot y \mid y \in Y\} \leq \inf \{z \cdot x \mid x \in X\}$.

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