Cournot competition

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Supplementary material for Knowledge, belief and rationality
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Two firms $i \in \{1,2\}$ simultaneously choose quantities $q_i \in Q_i = \mathbb{R}_+$. The market price is determined by the inverse demand function:

$$P(s) = P_0 - (q_1 + q_2), \qquad a, b \in \mathbb{R}_{++}$$

The firms have constant marginal cost $k \in \mathbb{R}_{++}$. Profits are thus given by:

$$u_i(q) = q_i P(q) = -q_i^2 + q_i(P_0 - k - q_{-i})$$

Proposition 1 *The only rationalizable quantities are $Q_i = \dfrac{1}{3}(P_0 - k)$.*

Proof. Let $(Q_i^n)_{i\in I, n\in\mathbb{N}} \in \mathbb{R}_+^{I\times\mathbb{N}}$ be the elimination sequence that eliminates *all* strictly dominated strategies at each stage, and let $Q^\infty = \cap_{n\in\mathbb{N}} Q^n$ be the set of rationalizable strategy profiles. That is, $Q^1 = Q$, and for all $i \in I$ and all $n \in \mathbb{N}$:[2]

$$Q_i^{n+1} = \left\{ q_i \in Q_i^n \;\middle|\; \left(\exists q_{-i} \in Q_{-i}^n\right)\left(\forall q_i' \in Q_i^n\right)\left(u_i(q_i, q_{-i}) \geq u_i(q_i', q_{-i})\right) \right\}$$

I will show that $Q_i^\infty = \{(P_0 - k)/3\}$ for $i = 1,2$. For that purpose, let $x \in \mathbb{R}^{\mathbb{N}}$ be the sequence of real numbers defined by:

$$x_1 = \frac{1}{2} \qquad \wedge \qquad \forall n \in \mathbb{N}:\; x_{n+1} = \frac{1}{2}\left(1 - x_n\right)$$

It can be easily shown (for instance by induction) that $x_n = -\sum_{i=1}^n (-2)^{-i}$ and, consequently:

$$\lim x_n = \lim \sum_{i=1}^n 2^{-2i+1} - \lim \sum_{i=1}^n 2^{-2i} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

Now let $N, M \subseteq \mathbb{N}$ be the sets of natural numbers given by:

$$N = \left\{ n \in \mathbb{N} \;\middle|\; Q^{2n-1} \subseteq \left[0, x_{2n-1}(P_0 - k)\right]^2 \right\}$$
$$M = \left\{ n \in \mathbb{N} \;\middle|\; Q^{2n} \subseteq \left[x_{2n}(P_0 - k), \infty\right)^2 \right\}$$

I will show by induction that $N = M = \mathbb{N}$. Notice that for all i and q_{-i}, u_i as a function of s_i is a strictly concave quadratic function, and it is strictly increasing for all $q_i < \mathrm{BR}_i(q_{-i})$ and strictly decreasing for all $q_i > \mathrm{BR}_i(q_{-i})$, where $\mathrm{BR}_i : \mathbb{R}_+ \to \mathbb{R}$ is given by:

$$\mathrm{BR}_i(q_{-i}) = \frac{1}{2}(P_0 - k - q_{-i})$$

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[2] Since the utility functions are linear, it suffices to consider degenerate beliefs. That is, a strategy is a best response if and only if it is a best response to a pure strategy.

Since $Q_{-i} = \mathbb{R}_+$, For all $q_{-i} \in Q_{-i}$ we have:

$$\mathrm{BR}_i(q_{-i}) \leq \frac{1}{2}(P_0 - k)$$

Consequently u_i is always decreasing for $q_i > (P_0 - k)/2$, and any such strategy is strictly dominated (by $q_i - \varepsilon$ with $\varepsilon > 0$ small enough). Therefore $Q_i^1 \subseteq [0, (P_0 - k)/2]$, that is, $1 \in N$ and $N \neq \emptyset$.

Now let $n \in N$ be an arbitrary point in N. For all $q_{-i} \in Q_{-i}^{2n-1}$ we have that $q_{-i} \leq x_{2n-1}(P_0 - k)$ and thus:

$$\mathrm{BR}_i(q_{-i}) \geq \frac{1}{2}(P_0 - k) - \frac{1}{2}\big(x_{2n-1}(P_0 - k)\big) = (P_0 - k)\left(\frac{1}{2}(1 - x_{2n-1})\right) = (P_0 - k)x_{2n}$$

Consequently u_i is always increasing for $q_i < x_{2n}(P_0 - k)$, and any such strategy is strictly dominated (by $q_i + \varepsilon$ with $\varepsilon > 0$ small enough). Therefore $Q_i^{2n} \subseteq [(x_{2n})(P_0 - k), \infty)$, that is, $n \in M$. Hence we know that $N \subseteq M$.

Furthermore, for all $q_{-i} \in Q_{-i}^{2n}$ we have that $q_{-i} \geq x_{2n}(P_0 - k)$ (since $n \in M$) and thus:

$$\mathrm{BR}_i(q_{-i}) \leq \frac{1}{2}(P_0 - k) - \frac{1}{2}\big(x_{2n}(P_0 - k)\big) = (P_0 - k)\left(\frac{1}{2}(1 - x_{2n})\right) = x_{2n+1}(P_0 - k)$$

Consequently u_i is always decreasing for $q_i > x_{2n+1}(P_0 - k)$ and any such strategy is strictly dominated. Therefore $Q_i^{2(n+1)-1} = Q_i^{2n+1} \subseteq [0, (x_{2n+1})(P_0 - k)]$, that is, $n + 1 \in N$. Therefore, by the induction principle $M = N = \mathbb{N}$, thus:

$$\begin{aligned} Q_i^{\infty} &\subseteq \bigcap \left\{ \big[x_{2n}(P_0 - k), x_{2n-1}(P_0 - k)\big] \;\middle|\; n \in \mathbb{N} \right\} \\ &= \big[\lim x_{2n}(P_0 - k), \lim x_{2n-1}(P_0 - k)\big] = \{(P_0 - k)/3\} \end{aligned}$$

where the first equality follows from the (easily shown) fact that x_{2n-1} and x_{2n} are monotonously decreasing and increasing respectively.

Finally, notice that:

$$\mathrm{BR}_i\left(\frac{P_0 - k}{3}\right) = \frac{1}{2}(P_0 - k) - \frac{1}{2} \cdot \frac{P_0 - k}{3} = \frac{P_0 - k}{3}$$

Therefore $(P_0 - k)/3$ is i's best response to $q_{-i} = (P_0 - k)/3$. Thus it is rationalizable and it is never eliminated. Hence $Q_i^{\infty} = \{(P_0 - k)/3\}$ ■