Knowledge and common knowledge

Slides 3 – Equilibrium

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n-firm Cournot competition

Rationalizablity

- n firms choose quantities $Q_i \in \mathbb{R}_+$
- Price is determined by the market and payoffs are:

$$u_i(Q) = Q_i \left(P_0 - k - \sum_{j=1}^{n} Q_j \right)$$

- If there is only one firm (monopoly), the only rational action is

$$Q_i^M = \frac{1}{2}(P_0 - k)$$

- If there are two firms (duopoly), the only rationalizable action is

$$Q_i^D = \frac{1}{3}(P_0 - k)$$

- If $n \geq 2$ then any action in $\left[0, Q^M\right]$ is rationalizable!

n-firm Cournot competition

Equilibrium

Proposition

In the unique Nash equilibrium every firm chooses $Q_i = \dfrac{1}{n+1}(P_0 - k)$

- Firm i's best response is given by

$$\mathrm{BR}_i(Q_{-i}) = \frac{1}{2}\left(P_0 - k - \textstyle\sum_{j \neq i} Q_j\right) \tag{1}$$

- Since best responses are single valued, players cannot be indifferent, and therefore every NE is in pure strategies
- Let $Q^* \in \mathbb{R}^n_+$ be a NE, for i we have $Q^*_i = \mathrm{BR}_i(Q^*_{-i})$ and thus from (1):

$$Q^*_i = P_0 - k - \textstyle\sum_{j \in I} Q^*_j \tag{2}$$

- This implies $Q^*_i = Q^*_j$ and hence $\sum_{j \in I} Q^*_j = nQ^*_i$
- From (2) this implies

$$Q^*_i = P_0 - k - nQ^*_i \quad \Rightarrow \quad Q^*_i = \frac{1}{n+1}(P_0 - k)$$

$\mathscr{QED}$

Example: Meeting in NY

	CP	GCS
CP	1 , 1	0 , 0
GCS	0 , 0	1 , 1

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Mixed strategies

- A mixed strategy for player i is a distribution $\sigma_i \in \Delta(S_i)$
- A mixed strategy profile is a vector of mixed strategies $\sigma = (\sigma_i)_{i \in I}$
- We use the notation $\sigma_{-i} = (\sigma_j)_{j \neq i}$ and $\sigma = (\sigma_i, \sigma_{-i})$
- The probability of strategy profile s according to σ is

$$\sigma(s) = \prod_{i \in I} \sigma_i(s_i)$$

- There are (at least) two possible interpretations:
  - Randomization – $\sigma_i(s_i)$ is the probability that player i chooses s_i
  - Conjectures – $\sigma_{-i}(s_{-i})$ is i's conjecture about $-i$'s choices

Nash equilibrium

- A Nash equilibrium is a mixed strategy profile σ^* for which *there are no profitable unilateral deviations*

- i.e. such that for every player i and every strategy $s_i \in S_i$ such that $\sigma_i(s_i) > 0$ and every potential deviation $s_i' \in S_i$:

$$\sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) u_i(s_i', s_{-i})$$

- In a NE, each player must be indifferent between all the strategies he/she chooses with positive probability
- This fact can be used to compute NE

Example: matching pennies

	H	T
H	$1\,,\ -1$	$-1\,,\ 1$
T	$-1\,,\ 1$	$1\,,\ -1$

- There are no NE in pure strategies
- For player 1 to be willing to randomize, it must be the case that

$$\begin{aligned}\mathbb{E}[\,u_1(H, s_2)\,] &= \sigma_2(H)(1) + \sigma_2(T)(-1) \\ &= \sigma_2(H)(-1) + \sigma_2(T)(1) = \mathbb{E}[\,u_1(T, s_{-i})\,]\end{aligned}$$

- Which implies $\sigma_2(H) = \sigma_2(T) = 1/2$
- A similar argument yields $\sigma_1(H) = \sigma_1(T) = 1/2$
- Hence the unique NE is $\big((1/2, 1/2),\ (1/2, 1/2)\big)$

Observations

- Every finite game has at least one NE (also every compact game with continuous payoffs)
- Strategies which played in NE are rationalizable
- If the game is dominance solvable, then the surviving strategy profile is a NE
- Generic finite games have an odd number of equilibria
- Finding all the NE of a game is a difficult computational problem

Example: a game of chicken

	W	S
W	4 , 4	1 , 5
S	5 , 1	0 , 0

- The game has two NE in pure strategies, (W, S) and (S, W)
- It also has a mixed NE $\sigma^* = \big((1/2, 1/2),\ (1/2, 1/2)\big)$
- The corresponding expected payoffs are:

$$U(W, S) = (1, 5) \qquad U(W, S) = (5, 1) \qquad U(\sigma^*) = (2.5, 2.5)$$

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Motivation

> *"If there is intelligent life on other planets, in a majority of them, they would have discovered correlated equilibrium before Nash equilibrium"*
>
> *– Roger Myerson*

- NE is defined in terms of mixed strategy profiles
- The resulting distribution over strategy profiles is a product distribution $\sigma(s) = \prod_{i \in I} \sigma_i(s_i)$
- Is there any good reason to impose the restriction that randomization should be independent?
- If we think about explicit randomization, could players not condition their choices on correlated random variables?
- If we think about conjectures, could i's beliefs about the choices of is opponents not be correlated?

Mediators

- Suppose that players hire a mediator to manage the game through correlated private recommendations
- The mediator chooses a strategy profile s according to a pre-specified distribution $\sigma \in \Delta(S)$, and privately informs each player about the strategy s_i that he/she is supposed to choose

- The marginal distributions are given by $\sigma_i(s_i) = \sum_{s_{-i} \in S_{-i}} \sigma(s_i, s_{-i})$
- σ does not have to be a product distribution! it may be the case that $\sigma(s) \neq \prod_i \sigma_i(s_i)$, where σ_i is the marginal distribution

- Each player is informed only about his recommendations, hence for s_i such that $\sigma(s_i) > 0$:

$$p_i(s_{-i}|s_i) = \sigma(s_{-i}|s_i) = \frac{\sigma(s_i, s_{-i})}{\sigma_i(s_i)}$$

Correlated equilibrium

- A distribution σ is a correlated equilibrium if and only if following recommendations is a Nash equilibrium of the corresponding mediated game
- σ is a correlated equilibrium if and only if for every s_i such that $\sigma_i(s_i) > 0$ we have

$$\sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) \cdot \sigma(s_{-i}|s_i) \geq \sum_{s_{-i} \in S_{-i}} u_i(s_i', s_{-i}) \cdot \sigma(s_{-i}|s_i)$$

- If we require $\sigma(s) = \prod_{i \in I} \sigma_i(s_i)$ then we obtain the definition of NE

Example: a game of chicken

Correlated strategies

	W	S
W	4 , 4	1 , 5
S	5 , 1	0 , 0

	W	S
W	θ_1	θ_2
S	θ_3	θ_4

- Distributions over S can be described by vectors $\theta = (\theta_n)_{n=1}^4 \in \mathbb{R}^n$ with:

$$\sum_{n=1}^{4} \theta_n = 1 \quad \text{and} \quad \theta_n \geq 0 \quad \text{for} \quad n = 1, 2, 3, 4$$

Example: a game of chicken

Marginal and conditional distributions

	W	S
W	4 , 4	1 , 5
S	5 , 1	0 , 0

	W	S
W	θ_1	θ_2
S	θ_3	θ_4

- Marginal distributions are given by:

$$\sigma_1(W) = \theta_1 + \theta_2 \qquad \sigma_1(S) = \theta_3 + \theta_4$$
$$\sigma_2(W) = \theta_1 + \theta_3 \qquad \sigma_2(S) = \theta_2 + \theta_4$$

- Assuming $\sigma_1(W) > 0$ and $\sigma_1(S) > 0$, the distributions over s_2 conditional on s_1 are given by:

$$\sigma(s_2 = W | s_1 = W) = \frac{\theta_1}{\theta_1 + \theta_2} \qquad \sigma(s_2 = S | s_1 = W) = \frac{\theta_2}{\theta_1 + \theta_2}$$
$$\sigma(s_2 = W | s_1 = S) = \frac{\theta_3}{\theta_3 + \theta_4} \qquad \sigma(s_2 = S | s_1 = S) = \frac{\theta_4}{\theta_3 + \theta_4}$$

Example: a game of chicken

Incentive constraints for the row player

	W	S
W	4 , 4	1 , 5
S	5 , 1	0 , 0

	W	S
W	θ_1	θ_2
S	θ_3	θ_4

- If $\theta_1 + \theta_2 > 0$, it must be the case that:

$$4\left(\frac{\theta_1}{\theta_1+\theta_2}\right) + 1\left(\frac{\theta_2}{\theta_1+\theta_2}\right) \geq 5\left(\frac{\theta_1}{\theta_1+\theta_2}\right) + 0\left(\frac{\theta_2}{\theta_1+\theta_2}\right)$$

- Which is equivalent to $\theta_2 \geq \theta_1$

- If $\theta_3 + \theta_4 > 0$, it must be the case that:

$$5\left(\frac{\theta_3}{\theta_3+\theta_4}\right) + 0\left(\frac{\theta_4}{\theta_3+\theta_4}\right) \geq 4\left(\frac{\theta_3}{\theta_3+\theta_4}\right) + 1\left(\frac{\theta_4}{\theta_3+\theta_4}\right)$$

- Which is equivalent to $\theta_3 \geq \theta_4$

Example: a game of chicken

Incentive constraints for the column player

	W	S
W	4 , 4	1 , 5
S	5 , 1	0 , 0

	W	S
W	θ_1	θ_2
S	θ_3	θ_4

- If $\theta_1 + \theta_3 > 0$, it must be the case that:

$$4\left(\frac{\theta_1}{\theta_1+\theta_3}\right) + 1\left(\frac{\theta_3}{\theta_1+\theta_3}\right) \geq 5\left(\frac{\theta_1}{\theta_1+\theta_3}\right) + 0\left(\frac{\theta_3}{\theta_1+\theta_3}\right)$$

- Which is equivalent to $\theta_3 \geq \theta_1$

- If $\theta_3 + \theta_4 > 0$, it must be the case that:

$$5\left(\frac{\theta_2}{\theta_2+\theta_4}\right) + 0\left(\frac{\theta_4}{\theta_2+\theta_4}\right) \geq 4\left(\frac{\theta_2}{\theta_2+\theta_4}\right) + 1\left(\frac{\theta_4}{\theta_2+\theta_4}\right)$$

- Which is equivalent to $\theta_2 \geq \theta_4$

Example: a game of chicken

Correlated equilibria

	W	S
W	4 , 4	1 , 5
S	5 , 1	0 , 0

	W	S
W	θ_1	θ_2
S	θ_3	θ_4

- A vector θ represents a CE if and only if

$$\theta_2, \theta_3 \geq \theta_1, \theta_4 \qquad \theta_n \geq 0 \qquad \sum_n \theta_n = 1$$

Example: a game of chicken

Correlated vs. Nash equilibria

	W	S
W	1/3	1/3
S	1/3	0

Best symmetric CE

	W	S
W	0	1/3
S	1/3	1/3

Worst symmetric CE

	W	S
W	0	1
S	0	0

Pure NE

	W	S
W	0	0
S	1	0

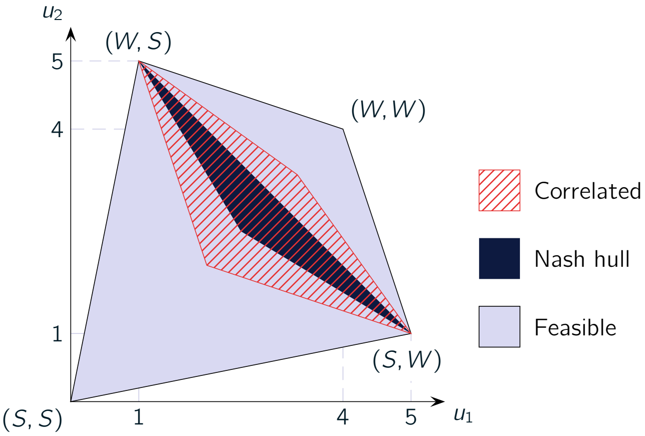
Pure NE

	W	S
W	1/4	1/4
S	1/4	1/4

Mixed NE

Example: a game of chicken

Correlated vs. Nash equilibria



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1. Motivation
2. Equilibrium
   - Nash equilibrium
   - Correlated equilibrium
3. Epistemic conditions
   - Common priors
   - On belief consistency

Nash equilibrium in pure strategies

Proposition

A pure strategy profile is a Nash equilibrium if and only if it can be chosen in some state in which all players are rational and choices are mutual knowledge.

- The proof is trivial because it suffices to consider single-state models
- Which makes the knowledge state uninteresting

- Rationality + MK of conjectures yields MK of rationality
- However, there are no explicit common knowledge assumptions

Nash equilibrium in 2-player games

Proposition

In two player games, a strategy profile is a Nash equilibrium if and only if it coincides with first order beliefs in some state in which there is mutual knowledge of rationality and conjectures.

[$\Leftarrow$]

- Consider some mixed strategy profile σ
- Let that ω^* be such that $\omega^* \in \mathrm{K_I}(\mathrm{RAT}) \subseteq \mathrm{RAT}$ and $\mathrm{K_I}(q_i(s_i|\omega) = \sigma_{-i})$ for $i = 1, 2$
- This implies that for i and $\omega \in \pi_i(\omega^*)$: $q_{-i}(\cdot|\omega) = \sigma_i$ and $c_{-i}(\omega) \in \mathrm{BR}_{-i}(q_{-i}(\cdot|\omega))$
- Hence every choice made by $-i$ in $\pi_i(\omega^*)$ is a best response to σ_i
- Since $q_i(\pi_i(\omega^*)|\omega^*) = 1$ and $q_i(\cdot|\omega^*) = \sigma_{-i}$, this implies that s_{-i} is a best response to σ_i whenever $\sigma_{-1}(s_{-i}) > 0$
- Which means that σ is a NE

Nash equilibrium in 2-player games

Proposition

In two player games, a strategy profile is a Nash equilibrium if and only if it coincides with first order beliefs in some state in which there is mutual knowledge of rationality and conjectures.

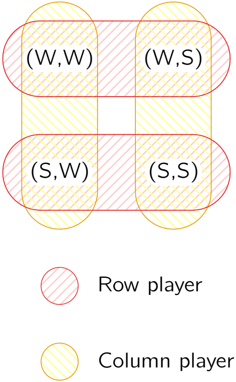
[⇒]

- Consider a NE σ^* and let $T_i = \{s_i \in S_i \mid \sigma_i(s_i) > 0\}$, $T = \times_{i \in I} T_i$
- Consider the model (Ω, Π, p, q, c) with:
  - $\Omega = T$, $c_i(\omega) = \omega$
  - $\Pi_i = \big\{\{s_i\} \times T_{-i} \,\big|\, s_i \in \mathrm{R}_i\big\}$
  - $p_i(s) = \sigma^*(s)$
  - $q_i\big(s_{-i} | \{s_i\} \times T_{-i}\big) = \sigma_{-i}(s_{-i})$
- By construction $\mathrm{RAT} = \Omega$ and conjectures are constant
- Hence there is mutual knowledge of rationality and conjectures

QED

Example: a game of chicken

Epistemic model for mixed NE



$\Omega = S$

$c(\omega) = \omega$

$p_i(\omega) = 1/4$

$\Pi_1 = \left\{\{s_1\} \times \mathrm{R}_2 \;\middle|\; s_1 \in \mathrm{R}_1\right\}$

$\Pi_2 = \left\{\{s_2\} \times \mathrm{R}_1 \;\middle|\; s_2 \in \mathrm{R}_2\right\}$

$\mathrm{RAT} = \mathrm{K}_{\mathrm{I}}^{\infty}(\mathrm{RAT}) = \Omega$

Example: a $2 \times 2 \times 2$ game

	a	b
a	1 , 1 , 0	1 , 0 , 0
b	0 , 1 , 0	0 , 0 , 0

A

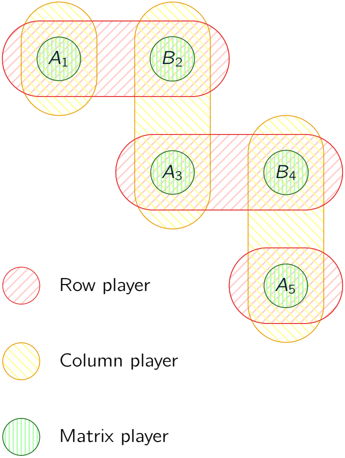
	a	b
a	-2 , -1 , 0	-2 , 0 , 0
b	0 , -1 , 0	0 , 0 , 0

B

- Player 3 is always indifferent between A and B
- Players 1 wants to choose a if $\Pr(A) > 2/3$ and b otherwise
- Players 2 wants to choose a if $\Pr(A) > 1/2$ and b otherwise
- There is no NE in which 2 chooses b with positive probability, and 1 chooses a with positive probability

Example: a $2 \times 2 \times 2$ game

Epistemic model



$$\Omega = \{A_1, A_3, A_5, B_2, B_4\}$$

$$\Pi_3 = \big\{\{\omega\} \,|\, \omega \in \Omega\big\}$$

$$\Pi_2 = \big\{\{A_1\}, \{B_2, A_3\}, \{B_4, A_5\}\big\}$$

$$\Pi_1 = \big\{\{A_1, B_2\}, \{B_3, A_4\}, \{A_5\}\big\}$$

$$c_3(A_n) = A \qquad c_3(B_n) = B$$

$$c_2(A_1) = a \qquad c_2(\omega \neq A_1) = b$$

$$c_1(\omega) = a$$

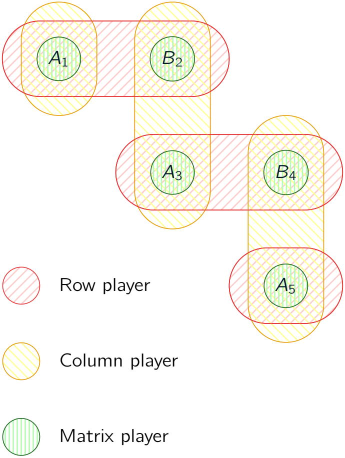
$$p_i(A_1) = 4/10 \qquad p_i(B_2) = 2/10$$

$$p_i(A_3) = 2/10 \qquad p_i(B_3) = 1/10$$

$$p_i(A_5) = 1/10$$

Example: a $2 \times 2 \times 2$ game

Mutual knowledge of conjectures and rationality



$$q_1\left(A|\{A_1, B_2\}\right) = 2/3 \qquad c_1\left(\{A_1, B_2\}\right) = a$$

$$q_1\left(A|\{A_3, B_4\}\right) = 2/3 \qquad c_1\left(\{A_3, B_4\}\right) = a$$

$$q_1\left(A|\{A_5\}\right) = 1 \qquad c_1\left(\{A_5\}\right) = a$$

$$\mathrm{RAT}_1 = \Omega$$

$$q_2\left(A|\{A_1\}\right) = 1 \qquad c_2\left(\{A_1\}\right) = a$$

$$q_2\left(A|\{B_2, A_3\}\right) = 1/2 \qquad c_2\left(\{B_2, A_3\}\right) = b$$

$$q_2\left(A|\{B_4, A_5\}\right) = 1/2 \qquad c_2\left(\{B_4, A_5\}\right) = b$$

$$\mathrm{RAT}_2 = \Omega$$

$$\mathrm{RAT} = \Omega \quad \Rightarrow \quad A_3 \in \mathrm{K}_{\mathrm{I}}^{\infty}(\mathrm{RAT})$$

$$A_3 \in \mathrm{K}_{\mathrm{I}}\left(q_1(A|w) = 2/3 \ \wedge \ q_2(A|w) = 1/2\right)$$

$$c(A_3) = (a, b, A)$$

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The common prior assumption

- Common knowledge or rationality only takes us as far as rationalizability
- Mutual knowledge of conjectures guarantees equilibrium, but only in some special cases

- In order to guaratee equilibrium in general we need the common prior assumption

$$p \equiv p_1 = p_2 = \ldots = p_n$$

- Under CP, we can interpret priors as objective probabilities, and think of the distribution $\sigma \in \Delta(S)$ induced by the model:

$$\sigma(s) = p\Big(\{\omega \in \Omega \mid c(\omega) = s\} \Big)$$

Correlated equilibrium

Proposition (Aumann, 1987)

A correlated strategy is a correlated equilibrium if and only if it can be induced by a model with common priors assumption and common knowledge of rationality.

[$\Leftarrow$]

- Consider a model with $\Omega = \text{RAT}$ and a common prior p, and let $\sigma \in \Delta(S)$ be given by $\sigma(s) = p(\{\omega \in \Omega \mid c(\omega) = s\})$
- For $s_i \in S_i$ let $E(s_i) = \{\omega \in \Omega \mid c_i(\omega) = s_i\}$
- For $s_i, s_i' \in S_i$

$$
\begin{aligned}
& \omega \in E(s_i): \quad \textstyle\sum_{s_{-i} \in S_{-i}} q_i(s_{-i}|\omega)\Big[u_i(s_i, s_{-i}) - u_i(s_i', s_{-i})\Big] \geq 0 \\
\Rightarrow \quad & \omega \in E(s_i): \quad \textstyle\sum_{s_{-i} \in S_{-i}} p(\omega) q_i(s_{-i}|\omega)\Big[u_i(s_i, s_{-i}) - u_i(s_i', s_{-i})\Big] \geq 0 \\
\Rightarrow \quad & \textstyle\sum_{\omega \in E(s_i)} \sum_{s_{-i} \in S_{-i}} p\big(\omega \cap E(s_{-i})\big)\Big[u_i(s_i, s_{-i}) - u_i(s_i', s_{-i})\Big] \geq 0 \\
\Rightarrow \quad & \textstyle\sum_{s_{-i} \in S_{-i}} p\big(E(s_i) \cap E(s_{-i})\big)\Big[u_i(s_i, s_{-i}) - u_i(s_i', s_{-i})\Big] \geq 0 \\
\Rightarrow \quad & \textstyle\sum_{s_{-i} \in S_{-i}} \sigma(s)\Big[u_i(s_i, s_{-i}) - u_i(s_i', s_{-i})\Big] \geq 0
\end{aligned}
$$

$\mathscr{QED}$

Correlated equilibrium

Proposition (Aumann, 1987)

A correlated strategy is a correlated equilibrium if and only if it can be induced by a model with common priors assumption and common knowledge of rationality.

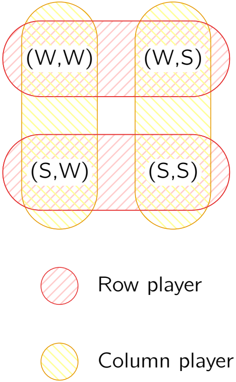
[⇒]

- Consider a CE σ^* and let $T_i = \{s_i \in S_i \mid \sigma_i(s_i) > 0\}$, $T = \times_{i \in I} T_i$
- Consider the model (Ω, Π, p, q, c) with:
  - $\Omega = T$, $c_i(\omega) = \omega$
  - $\Pi_i = \big\{\{s_i\} \times T_{-i} \;\big|\; s_i \in \mathrm{R}_i\big\}$
  - $p_i(s) = \sigma^*(s)$
  - $q_i\big(s_{-i}|\{s_i\} \times T_{-i}\big) = \sigma(s_{-i}|s_i)$
- By construction $\mathrm{RAT} = \Omega$ and there is a common prior

QED

Example: a game of chicken

Epistemic model for CE



$$\Omega = S$$

$$c(\omega) = \omega$$

$$p_i(W, W) = p_i(W, S) = p_i(S, W) = 1/3$$

$$p_i(S, S) = 0$$

$$\Pi_1 = \left\{ \{s_1\} \times \mathrm{R}_2 \,\middle|\, s_1 \in \mathrm{R}_1 \right\}$$

$$\Pi_2 = \left\{ \{s_2\} \times \mathrm{R}_1 \,\middle|\, s_2 \in \mathrm{R}_2 \right\}$$

$$\mathrm{RAT} = \mathrm{K}_{\mathrm{I}}^{\infty}(\mathrm{RAT}) = \Omega$$

Nash equilibrium

Proposition

A mixed strategy profile is a Nash equilibrium if and only if it can be induced by a model with common priors, and common knowledge of conjectures [and common knowledge of rationality].

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Summary of epistemic conditions

Solution concept	**Rationality**	**Choices/conjectures**	**Priors**
Rationalizability	common knowledge	–	–
NE in pure strategies	fact	mutual knowledge	–
NE in 2–player games	mutual knowledge	mutual knowledge	–
Nash Equilibrium	common knowledge	common knowledge	common priors
Correlated Equilibrium	common knowledge	–	common priors

How can agreement be reached?

- To move from ratonalizability to equilibrium we need either agreement on conjectures (2PNE) or agreement on prior beliefs
- How can this agreement be achieved?

- Rational may have common priors
- Chance – Blundering into equilibrium
- Focal points
- Communication/mediation
- Precedence – adaptation, learning, or evolution

A final thought

"Game theory... is deficient to the extent it assumes other features to be common knowledge, such as one player's probability assessment about another's preferences or information. I foresee the progress of game theory as depending on successive reductions in the base of common knowledge required to conduct useful analyses of practical problems. Only by repeated weakening of common knowledge assumption will the theory approximate reality."

Robert Wilson (1987)

Thanks

This concludes the course!

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